

CUNTZ-KRIEGER UNIQUENESS THEOREM FOR CROSSED PRODUCTS BY HILBERT BIMODULES

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ABSTRACT. It is proved that a C^* -algebra generated by any faithful covariant representation of a Hilbert bimodule X is canonically isomorphic to the crossed product $\mathcal{A} \rtimes_X \mathbb{Z}$ provided that the action of Rieffel's induced representation functor is topologically free. It is discussed how this result could be applied to universal C^* -algebras generated by relations with a circle gauge action. In particular, it leads to generalizations of isomorphism theorems for various crossed products, and is shown to be equivalent to Cuntz-Krieger uniqueness theorem for finite graph C^* -algebras (on that occasion an intriguing realization of Cuntz-Krieger algebras as crossed products by Exel's interactions is discovered).

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INTRODUCTION

The *problem of uniqueness* discussed in the paper is related to the origins of the C^* -algebra theory and particularly the theory of universal C^* -algebras generated by objects that satisfy prescribed relations. The first important examples that gave a strong impetus for the development of such a theory are algebras generated by quantum anti-commutation relations and algebras generated by canonical commutation relations. The great advantage of relations of CAR and CCR type is a *uniqueness of representation* – the C^* -algebras generated by such relations are defined uniquely up to isomorphism preserving the relations, see e.g. [Sla71]. On the other hand there are many important relations that do not possess this uniqueness property, and among the most remarkable ones are the Cuntz-Krieger relations:

$$(1) \quad S_i^* S_i = \sum_{j=1}^n A(i, j) S_j S_j^*, \quad S_i^* S_j = \delta_{i,j} S_i^* S_i, \quad i = 1, \dots, n,$$

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where $\{A(i, j)\}$ is a given $n \times n$ zero-one matrix and $\delta_{i,j}$ is Kronecker symbol. In [CK80] J. Cuntz and W. Krieger formulated the co-called *condition (I)* which is necessary and sufficient for the relations (1) to have the uniqueness property. Since then similar results are called *Cuntz-Krieger uniqueness theorems*. In particular, to cover the situation of infinite graphs condition (I) was replaced in [KPR98] by *condition (L)* which naturally carries over to topological graphs and lead T. Katsura to the result [Kat04', Thm. 5.12] that contains as special cases both the Cuntz-Krieger uniqueness theorem for graph C^* -algebras and isomorphism theorem for homeomorphism C^* -algebras (crossed products of commutative algebras by automorphisms). In general, the so-called *isomorphism theorems* for crossed products present conditions called *topological* (or *metrical*) *freeness* which imply that every covariant representation of a dynamical system generate an isomorphic copy of the associated crossed product (see [AL94, pp. 225-226] for a brief survey of such results). Plainly, both Cuntz-Krieger uniqueness and isomorphism theorems present special instances of a general uniqueness property and thus they provoke a question of existence and form of a general uniqueness theorem. The present paper is a step towards such a result.

It is well known, see e.g. [Kat03], [MS98], [FMR03], that C^* -algebras associated with various structures, including graph C^* -algebras and crossed products by endomorphisms, can be modeled and investigated onto the general ground of the so-called relative Cuntz-Pimsner algebras (theory of which was initiated by M. Pimsner [Pim97]). Actually, by [AEE98, Thm 3.1], any C^* -algebra \mathcal{B} equipped with a semi-saturated circle action γ can be naturally modeled as a relative Cuntz-Pimsner algebra; namely a crossed product by the first spectral subspace \mathcal{B}_1 for γ considered as a Hilbert C^* -bimodule over the fixed point C^* -algebra \mathcal{B}_0 . Literally, $\mathcal{B} = \mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}$. Thus we propose a two-step method of investigation universal C^* -algebras $C^*(\mathcal{G}, \mathcal{R})$ generated by \mathcal{G} subject to relations \mathcal{R} that admit a semi-saturated circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda \in \mathbb{T}}$ which schematically could be presented as follows:

$$(2) \quad \begin{array}{ccc} (\mathcal{G}, \mathcal{R}, \{\gamma_\lambda\}_{\lambda \in \mathbb{T}}) & \xrightarrow{\text{step 1}} & (\mathcal{B}_0, \mathcal{B}_1) \\ \text{relations, circle action} & \text{Hilbert bimodule} & \text{step 2} \\ & \text{(reversible dynamics)} & C^*(\mathcal{G}, \mathcal{R}) = \mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z} \\ & & \text{universal } C^*\text{-algebra} \end{array}$$

where \mathcal{B}_0 is the fixed point C^* -algebra and \mathcal{B}_1 is the first spectral subspace for γ .

More precisely, it is profitable to think of the Hilbert bimodule $(\mathcal{B}_0, \mathcal{B}_1)$ as a noncommutative reversible dynamical system associated to $(\mathcal{G}, \mathcal{R})$. We show that the induced representation functor $\mathcal{B}_1\text{-Ind}$ yields a partial homeomorphism \widehat{h} on the spectrum $\widehat{\mathcal{B}}_0$ of \mathcal{B}_0 , and if \widehat{h} is topologically free, then any faithful copy of the bimodule $(\mathcal{B}_0, \mathcal{B}_1)$ generates the C^* -algebra $C^*(\mathcal{B}_1, \mathcal{B}_0)$ naturally isomorphic to $\mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}$. Moreover, under the assumption that \widehat{h} is free we establish lattice isomorphism between \widehat{h} -invariant subsets $\widehat{\mathcal{B}}_0$ and ideals in $\mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}$, and give a simplicity criterion for $\mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}$. These results are presented in section 1 and solve the problem of uniqueness for crossed products by Hilbert bimodules. In particular, they completely clarify the step 2 in the scheme (2).

The necessary condition for $(\mathcal{G}, \mathcal{R})$ to have the uniqueness property is that any faithful representation of $(\mathcal{G}, \mathcal{R})$ give rise to a faithful representation of $(\mathcal{B}_0, \mathcal{B}_1)$ and this is in essence what the so-called gauge-uniqueness theorems state. If such a theorem holds, then $(\mathcal{G}, \mathcal{R})$ possess uniqueness property if and only if $(\mathcal{B}_0, \mathcal{B}_1)$ does.

Thus if one accomplishes step 1 in the scheme (2) and is able to find conditions in terms of $(\mathcal{G}, \mathcal{R})$ implying topological freeness of \widehat{h} , one gets a version of uniqueness theorem for $C^*(\mathcal{G}, \mathcal{R})$. Similarly, identifying \widehat{h} -invariant subsets of $\widehat{\mathcal{B}}_0$ in terms of $(\mathcal{G}, \mathcal{R})$ one obtains ideal lattice description and simplicity criteria for $C^*(\mathcal{G}, \mathcal{R})$. Obviously, rephrasing properties of \widehat{h} in terms of $(\mathcal{G}, \mathcal{R})$ is in general a very complex problem. However, as a rule investigation of algebras of type $C^*(\mathcal{G}, \mathcal{R})$, in either explicit or implicit way, involves or reduces to investigation of their core C^* -algebras, which means that a work has to be done anyway. Moreover, the explicit description of $(\mathcal{B}_0, \mathcal{B}_1)$ and understanding its dynamics has always great merit and sheds a new interesting light on the structure of $C^*(\mathcal{G}, \mathcal{R})$.

We present several concrete and significant applications to support the above point of view and to illustrate (2). We start in section 2 by considering partial isometric crossed products in a sense associated with a reversible (noncommutative) dynamics. Namely, we let $\mathcal{G} = \mathcal{A} \cup \{S\}$, where \mathcal{A} is a C^* -algebra and S is a partial isometry, and relations \mathcal{R} arise from a partial automorphism (θ, I, J) [Exe94, 3.1] or an interaction $(\mathcal{V}, \mathcal{H})$ [Exe07, Defn. 3.1]. In the case of interaction the algebras $C^*(\mathcal{G}, \mathcal{R})$ include various crossed products by endomorphisms with hereditary range, and as we show $C^*(\mathcal{G}, \mathcal{R})$ is isomorphic to what is suggested by the author of [Exe07] as a candidate for *crossed product by the interaction* $(\mathcal{V}, \mathcal{H})$, and we denote it by $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$. In general, we show that $\mathcal{A} = \mathcal{B}_0$ and $S\mathcal{A} = \mathcal{B}_1$, and hence dual maps $\widehat{\theta}$ or $\widehat{\mathcal{V}}$ (depending on the case) coincide with the inverse to the partial homeomorphism \widehat{h} implemented by \mathcal{B}_1 . This allows us to apply almost directly the results of section 1 to get the corresponding results for associated crossed products (the step 1 in the scheme (2) is not sophisticated).

The situation is quite different when the relations are somehow related to irreversible dynamics, since then step 1 in the scheme (2) is non-trivial even for not complicated systems $(\mathcal{G}, \mathcal{R})$. We discuss two situations exhibiting this phenomena but in two different ways. In both cases, we start in a sense from a commutative algebra \mathcal{A} . However, in the first case the initial dynamics is implemented by a multiplicative homomorphism and the corresponding reversible dynamical system $(\mathcal{B}_0, \mathcal{B}_1)$ is commutative. In the other case the initial dynamics is “barely” positive linear and we end up in a highly noncommutative system $(\mathcal{B}_0, \mathcal{B}_1)$.

Namely, in section 3 we treat the case where $\mathcal{G} = \mathcal{A} \cup \{S\}$ and \mathcal{R} is related to a non-surjective endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ of a unital commutative C^* -algebra $\mathcal{A} = C(M)$. Equivalently \mathcal{R} could be expressed in terms of a partial irreversible dynamical system (M, φ) where M is a compact Hausdorff space. Then

$$(3) \quad \mathcal{A} \subset \mathcal{B}_0, \quad \mathcal{A} \neq \mathcal{B}_0,$$

and the reason why $\mathcal{A} \neq \mathcal{B}_0$ is deeply related to irreversibility of the mapping φ . In particular, by the results of [Kwa’], [KL08] it is known that $\mathcal{B}_0 = C(\widetilde{M})$ is commutative and S generates on \widetilde{M} a partial homeomorphism $\widetilde{\varphi}$ such that $(\widetilde{M}, \widetilde{\varphi})$ is a natural reversible extension of (M, φ) . The system $(\widetilde{M}, \widetilde{\varphi})$ is dual to $(\mathcal{B}_0, \mathcal{B}_1)$ and has a complicated structure related to such topological and dynamical objects as *hyperbolic attractors*, *irreducible continua* or *systems associated with classical substitution tilings*. Since the complete description of $(\widetilde{M}, \widetilde{\varphi})$ in terms of (M, φ) is available, see [Kwa’], we may use it to identify the topological freeness of $\widetilde{\varphi}$ in terms

of φ . This leads us to the uniqueness theorem and description of ideal structure for the covariance C^* -algebra $C^*(M, \varphi) = C^*(\mathcal{G}, \mathcal{R})$.

We devote section 4 to Cuntz-Krieger algebras. Here the starting structure $(\mathcal{G}, \mathcal{R})$ is given by a finite directed graph $E = (E^0, E^1, r, s)$ or as it is indicated by a positive linear bounded operator $\mathcal{H} : \mathcal{A} \rightarrow \mathcal{A}$ acting on a finite dimensional commutative C^* -algebra

$$\mathcal{A} \cong \mathbb{C}^n.$$

In this case relations (3) also hold ($\mathcal{A} = \mathcal{B}_0$ if and only if the maps r, s are injective) and the structure of \mathcal{B}_0 is well known – it is the AF-algebra \mathcal{F}_E described in terms of a Bratteli diagram $\Lambda(E)$ constructed from E . In order to get a clear description of the action of \mathcal{B}_1 on $\mathcal{B}_0 = \mathcal{F}_E$ we show that the C^* -algebra $C^*(E) = C^*(\mathcal{G}, \mathcal{R})$ can be naturally realized as a crossed by an interaction $(\mathcal{V}, \mathcal{H})$ (probably, it is the *first non-trivial and significant example of such a crossed-product!*). We construct from the graph E two positive linear bounded operators $\mathcal{V}, \mathcal{H} : \mathcal{F}_E \rightarrow \mathcal{F}_E$ such that there is a natural gauge-invariant isomorphism $C^*(E) \cong \mathcal{F}_E \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$. Pictorially speaking, \mathcal{V} acts like a shift on the Bratteli diagram $\Lambda(E)$ and its "diagonalization" is a topological Markov chain (Ω_E, σ_E) , cf. subsection 4.3. We provide a description of the system $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$ dual to $(\mathcal{V}, \mathcal{H})$ that among the other things shows that topological freeness of the map $\widehat{\mathcal{V}}$ is equivalent to condition (L) for the graph E , and open $\widehat{\alpha}$ -invariant sets correspond to hereditary and saturated subsets of E^0 . Thus our general approach applied to graph algebras leads to the classical results of J. Cuntz, W. Krieger [CK80], [Cun81], and their successors [KPR98], [KPRR97], [BPRS00]. We feel, however, that the main value of the above development rests in a discovery of new dynamical properties of graphs that recover intriguing relationships between stochastic, operator theoretical and geometrical nature of the objects considered.

We also note an interesting, different, axiomatic approach to the uniqueness problem developed by Burgstaller [Bur06]. The main difference is that axioms introduced in [Bur06] require that the fixed point algebra \mathcal{B}_0 is approximately finite and hence they do not apply to general crossed products.

0.1. Background and notation. By \mathcal{A}, \mathcal{B} , etc. we denote C^* -algebras and we adhere to the convention that $\beta(\mathcal{A}, \mathcal{B}) = \overline{\text{span}}\{\beta(a, b) \in \mathcal{D} \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ for maps $\beta : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$ such as inner products, multiplications or representations. By a homomorphism, epimorphism, etc. we always mean an involution preserving map. All the ideals in C^* -algebras are assumed to be closed and two sided. We let the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ start from zero.

A *right Hilbert \mathcal{A} -module* X is a Banach space X together with a right action of \mathcal{A} on X and an \mathcal{A} -valued inner product being a sesqui-linear form $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ satisfying

$$\langle x, ya \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} a, \quad \langle x, y \rangle_{\mathcal{A}}^* = \langle y, x \rangle_{\mathcal{A}}, \quad \langle x, x \rangle_{\mathcal{A}} \geq 0, \quad \|x\| = \|\langle x, x \rangle_{\mathcal{A}}\|^{1/2},$$

for all $x, y \in X, a \in \mathcal{A}$. Similarly, one defines a left Hilbert \mathcal{A} -module. In particular, if X is a right (resp. left) Hilbert \mathcal{A} module we denote by \widetilde{X} the left (resp. right) *Hilbert \mathcal{A} -module dual* to X (\widetilde{X} is anti-linearly isomorphic to X). In the case X is both a left Hilbert \mathcal{B} -module and a right Hilbert \mathcal{A} -module with respective inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and ${}_{\mathcal{B}}\langle \cdot, \cdot \rangle$ satisfying the so-called *imprimitivity condition*:

$$(4) \quad x \cdot \langle y, z \rangle_{\mathcal{A}} = {}_{\mathcal{B}}\langle x, y \rangle \cdot z, \quad \text{for all } x, y, z \in X,$$

then X is called a *Hilbert \mathcal{B} - \mathcal{A} -bimodule*, cf. [BMS94, 1.8]. In particular, if a Hilbert \mathcal{B} - \mathcal{A} -bimodule X is full, that is if ${}_B\langle X, X \rangle = \mathcal{B}$ and $\langle X, X \rangle_{\mathcal{A}} = \mathcal{A}$, then X is an *imprimitivity bimodule* and algebras \mathcal{A} and \mathcal{B} are said to be *Morita equivalent* (as a general reference concerning Hilbert C^* -modules and related objects we recommend [RW98])

Suppose now that X is a right Hilbert \mathcal{A} -module X equipped with a left action of \mathcal{B} such that $\langle bx, y \rangle_{\mathcal{A}} = \langle x, b^*y \rangle_{\mathcal{A}}$, $b \in \mathcal{B}$, $x, y \in X$ (in the case $\mathcal{A} = \mathcal{B}$, X is called a *C^* -correspondence* over \mathcal{A}). For any right Hilbert \mathcal{B} -module Y there is a naturally defined tensor product right Hilbert \mathcal{A} -module $Y \otimes X$ where $\langle y_1 \otimes x_1, y_2 \otimes x_2 \rangle_{\mathcal{A}} = \langle x_1, \langle y_1, y_2 \rangle_{\mathcal{B}} x_2 \rangle_{\mathcal{A}}$, $x_i \in X$, $y_i \in Y$. Similarly, for a representation $\pi : \mathcal{A} \rightarrow L(H)$ into the algebra of all linear bounded operators in a Hilbert space H there is a well defined Hilbert space $X \otimes_{\pi} H$ generated by simple tensors $x \otimes_{\pi} h$, $x \in X$, $h \in H$, satisfying

$$(5) \quad \langle x_1 \otimes_{\pi} h_1, x_2 \otimes_{\pi} h_2 \rangle_{\mathbb{C}} = \langle h_1, \pi(\langle x_1, x_2 \rangle_{\mathcal{A}}) h_2 \rangle_{\mathbb{C}},$$

and the left action of \mathcal{B} on X defines (induces) via the formula

$$(6) \quad X\text{-Ind}(\pi)(b)(x \otimes_{\pi} h) = (bx) \otimes_{\pi} h$$

a representation $X\text{-Ind}(\pi) : \mathcal{B} \rightarrow L(X \otimes H)$ called an *induced representation*. The celebrated Rieffel's result, cf. [RW98, Cor. 3.33], states that if X is an imprimitivity \mathcal{B} - \mathcal{A} bimodule, then the induced representation functor $X\text{-Ind}$ factors through to the homeomorphism $X\text{-Ind} : \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{B}}$ between the spectra of algebras \mathcal{A} and \mathcal{B} .

By a circle action we mean an action $\gamma : \mathbb{T} \rightarrow \text{Aut}(\mathcal{B})$ of the group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ on a C^* -algebra \mathcal{B} which is point-wise continuous. For such an action and each $n \in \mathbb{Z}$ the formula

$$\mathcal{E}_n(b) := \int_{\mathbb{T}} \gamma_z(b) z^{-n} dz$$

defines a projection $\mathcal{E}_n : \mathcal{B} \rightarrow \mathcal{B}$, called *n -th spectral projection*, onto the subspace

$$\mathcal{B}_n := \{b \in \mathcal{B} : \gamma_z(b) = z^n b\}$$

called *n -th spectral subspace* for γ . Spectral subspaces specify a \mathbb{Z} -gradation on \mathcal{B} . Namely, $\bigoplus_{n \in \mathbb{Z}} \mathcal{B}_n$ is dense in \mathcal{B} , cf. e.g. [Exe94], and

$$\mathcal{B}_n \mathcal{B}_m \subset \mathcal{B}_{n+m}, \quad \mathcal{B}_n^* = \mathcal{B}_{-n} \quad \text{for all } n, m \in \mathbb{Z}.$$

In particular, \mathcal{B}_0 is a C^* -algebra – the fixed point algebra for γ , and $\mathcal{E}_0 : \mathcal{B} \rightarrow \mathcal{B}_0$ is a conditional expectation. Each spectral subspace \mathcal{B}_n , $n \in \mathbb{Z}$, is naturally equipped with the structure of the \mathcal{B}_0 -Hilbert bimodule where bimodule operations are inherited from \mathcal{B} and ${}_{{\mathcal{B}_0}}\langle x, y \rangle = xy^*$ and $\langle x, y \rangle_{{\mathcal{B}_0}} = x^*y$.

The action γ is called *saturated* if $\mathcal{B} = C^*(\mathcal{B}_1)$, that is if \mathcal{B} is generated by the first spectral subspace \mathcal{B}_1 , and γ is said to be *semi-saturated* if $\mathcal{B} = C^*(\mathcal{B}_0, \mathcal{B}_1)$. Alternatively, in terms of the Hilbert bimodule \mathcal{B}_1 , one can see that γ is semi-saturated if and only if we have a natural isomorphism $\mathcal{B}_1^{\otimes n} \cong \mathcal{B}_n$ for all $n = 1, 2, \dots$, cf. [Exe94, Prop. 4.8], and γ is saturated if additionally \mathcal{B}_1 is an imprimitivity bimodule.

The following result seems to be a part of a folklore for C^* -algebraists and the equivalence i) \Leftrightarrow iii) will be one of main tools in the present paper.

Theorem 0.1 (isomorphism theorems for C^* -algebras with circle actions). *Let $\Psi : \mathcal{B} \rightarrow \mathcal{B}'$ be an epimorphism of C^* -algebras where \mathcal{B} is equipped with a circle*

action γ and let $\mathcal{B}_0 = \{a \in C : \gamma_z(a) = a, z \in \mathbb{T}\}$ be the fixed point subalgebra of \mathcal{B} . The following conditions are equivalent

- i) Ψ is an isomorphism.
- ii) Ψ is injective on \mathcal{B}_0 and there exists a circle action γ' on \mathcal{B}' such that Ψ is gauge invariant, i.e. $\Psi \circ \gamma \equiv \gamma' \circ \Psi$.
- iii) for the conditional expectation $\mathcal{E}_0(a) = \int_{\mathbb{T}} \gamma_z(a) d\mu(z)$ onto \mathcal{B}_0 the following inequality holds

$$(7) \quad \|\mathcal{E}_0(a)\| \leq \|\Psi(a)\|, \quad \text{for all } a \in \mathcal{B}.$$

Proof. For the equivalence i) \Leftrightarrow ii) see for instance [Exe94, 2.9]. Implication i) \Rightarrow iii) is obvious. To see iii) \Rightarrow i) note that the images $\Psi(\mathcal{B}_n)$ of the spectral subspaces form an orthogonal sum $\bigoplus_{n \in \mathbb{Z}} \Psi(\mathcal{B}_n)$ dense in $\mathcal{B}' = \Psi(\mathcal{B})$. Indeed, if $\Psi(\sum_{n=-N}^N x_n) = 0$ where $x_n \in \mathcal{B}_n$, then for each $m = -N, \dots, N$

$$0 = \|\Psi\left(\sum_{n=-N}^N x_n\right)\Psi(x_m^*)\| \geq \|x_m x_m^*\|$$

that is $x_m = 0$ and consequently $\sum_{n=-N}^N x_n = 0$. Thus the result follows, for instance, from [DR87, Lem. 2.11]. \square

Equivalence i) \Leftrightarrow ii) is known as *gauge-uniqueness theorem*, and the inequality (7) is often called *property (*)*, cf. [AL94].

Suppose we are given an abstract set of generators \mathcal{G} and a set of $*$ -algebraic relations \mathcal{R} in a free non-unital $*$ -algebra \mathbb{F} generated by \mathcal{G} . A representation π of the pair $(\mathcal{G}, \mathcal{R})$ is the set of operators $\{\pi(g)\}_{g \in \mathcal{G}} \subset L(H)$ on a Hilbert space H satisfying the relations \mathcal{R} . Each such representation extends uniquely to a $*$ -homomorphism, also denoted by π , from \mathbb{F} into $L(H)$. The pair $(\mathcal{G}, \mathcal{R})$ is said to be *non-degenerate* if there is a representation $\{\pi(g)\}_{g \in \mathcal{G}} \subset L(H)$ of $(\mathcal{G}, \mathcal{R})$ which is *faithful* in the sense that $\pi(g) \neq 0$ for all $g \in \mathcal{G}$, and $(\mathcal{G}, \mathcal{R})$ is said to be *admissible* if the function $|||\cdot||| : \mathbb{F} \rightarrow [0, \infty]$ given by

$$|||w||| = \sup\{\|\pi(w)\| : \pi \text{ is a representation of } (\mathcal{G}, \mathcal{R})\}$$

is finite. In general, the set $\mathbb{I} := \{w \in \mathbb{F} : |||w||| = 0\}$ is a self-adjoint ideal in \mathbb{F} , and if $(\mathcal{G}, \mathcal{R})$ is non-degenerate, \mathbb{I} is the smallest self-adjoint ideal in \mathbb{F} such that the relations \mathcal{R} become valid in the quotient \mathbb{F}/\mathbb{I} . For an admissible pair $(\mathcal{G}, \mathcal{R})$, $|||\cdot|||$ is a C^* -seminorm, and we denote the completion of \mathbb{F}/\mathbb{I} under $|||\cdot|||$ by $C^*(\mathcal{G}, \mathcal{R})$ and call it a *universal C^* -algebra* generated by \mathcal{G} subject to relations \mathcal{R} , see [Bla85]. C^* -algebra $C^*(\mathcal{G}, \mathcal{R})$ is characterized by the property that any representation of $(\mathcal{G}, \mathcal{R})$ extends uniquely to a representation of $C^*(\mathcal{G}, \mathcal{R})$ and all representations of $C^*(\mathcal{G}, \mathcal{R})$ arise in that manner. A non-degenerate admissible pair $(\mathcal{G}, \mathcal{R})$ is said to have *uniqueness property* if any faithful representation of $(\mathcal{G}, \mathcal{R})$ extends to a faithful representation of $C^*(\mathcal{G}, \mathcal{R})$.

Let $(\mathcal{G}, \mathcal{R})$ be non-degenerate and admissible. There is a natural torus action $\{\gamma_\lambda\}_{\lambda \in \mathbb{T}^{\mathcal{G}}}$ on \mathbb{F} determined by the formula

$$\gamma_\lambda(g) = \lambda_g g, \quad \text{for } g \in \mathcal{G} \text{ and } \lambda = \{\lambda_h\}_{h \in \mathcal{G}} \in \mathbb{T}^{\mathcal{G}}.$$

If moreover there is a closed subgroup $H \subset \mathbb{T}^{\mathcal{G}}$ such that the action $\gamma = \{\gamma_\lambda\}_{\lambda \in H}$ leaves invariant the ideal \mathbb{I} , then it gives rise to an action on $C^*(\mathcal{G}, \mathcal{R})$. Actions that arise in that manner are called *gauge actions*. In particular, a circle gauge

action $\gamma = \{\gamma_\lambda\}_{\lambda \in \mathbb{T}}$ on $C^*(\mathcal{G}, \mathcal{R})$ is semi-saturated if and only if $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1$ for some disjoint sets $\mathcal{G}_0, \mathcal{G}_1$ and

$$\gamma_\lambda(g_0) = g_0, \quad \gamma_\lambda(g_1) = \lambda g_1, \quad \text{for all } g_i \in \mathcal{G}_i, i = 1, 2.$$

If $C^*(\mathcal{G}, \mathcal{R})$ is equipped with such a circle action, the necessary condition for $(\mathcal{G}, \mathcal{R})$ to possess uniqueness property is that each faithful representation of $(\mathcal{G}, \mathcal{R})$ give rise to a faithful representation of the fixed-point algebra $\mathcal{B}_0 \subset C^*(\mathcal{G}, \mathcal{R})$, and if this condition is fulfilled, then the equivalent conditions of Theorem 0.1 apply.

1. THE MAIN RESULT

Let \mathcal{A} be a C^* -algebra and X be an Hilbert \mathcal{A} - \mathcal{A} -bimodule, briefly a *Hilbert bimodule* over \mathcal{A} .

Definition 1.1 ([AEE98], Defn. 2.4). A *crossed product* $\mathcal{A} \rtimes_X \mathbb{Z}$ of \mathcal{A} by X is the universal C^* -algebra $C^*(\mathcal{G}, \mathcal{R})$ generated by the set $\mathcal{G} = \mathcal{A} \cup X$ subject to all algebraic relations \mathcal{R} defined in the pair (\mathcal{A}, X) . We identify representations of $(\mathcal{G}, \mathcal{R})$ with pairs $(\pi_{\mathcal{A}}, \pi_X)$ consisting of representations of \mathcal{A} and X into $L(H)$ such that all module operations become the ones inherited from $L(H)$, i.e.

$$(8) \quad \pi_X(ax) = \pi_{\mathcal{A}}(a)\pi_X(x), \quad \pi_X(xa) = \pi_X(x)\pi_{\mathcal{A}}(a),$$

$$(9) \quad \pi_{\mathcal{A}}(\langle x, y \rangle_{\mathcal{A}}) = \pi_X(x)^* \pi_X(y), \quad \pi_{\mathcal{A}}(\langle x, y \rangle) = \pi_X(x)\pi_X(y)^*,$$

for all $a \in \mathcal{A}, x, y \in X$. We call $(\pi_{\mathcal{A}}, \pi_X)$ a *covariant representation* of X , and we say that $(\pi_{\mathcal{A}}, \pi_X)$ is faithful if $\pi_{\mathcal{A}}$ is faithful (then π_X is automatically isometric). We denote by $(\pi_{\mathcal{A}} \rtimes \pi_X)$ the representation of $\mathcal{A} \rtimes_X \mathbb{Z}$ corresponding to $(\pi_{\mathcal{A}}, \pi_X)$ and call it an integrated form of $(\pi_{\mathcal{A}}, \pi_X)$.

For $n > 0$ we let $X^{\otimes n}$ be the n -th power tensor product of X and let \tilde{X} be the C^* -bimodule dual to X . We treat $X^{\otimes 0} = \mathcal{A}$ as a C^* -bimodule with the standard operations and for $n < 0$ we put $X^{\otimes n} = \tilde{X}^{\otimes |n|}$. In particular, if \flat denotes both anti-linear isomorphisms $X \rightarrow \tilde{X}$ and $X^{\otimes n} \rightarrow \widetilde{X^{\otimes n}}$, then $\flat(x_1) \otimes \dots \otimes \flat(x_n) \mapsto \flat(x_n \otimes \dots \otimes x_1)$ extends to the isomorphism $\tilde{X}^{\otimes n} \cong \widetilde{X^{\otimes n}}$. A covariant representation $(\pi_{\mathcal{A}}, \pi_X)$ of X yields for all $n \in \mathbb{Z}$ a covariant representations $(\pi_{\mathcal{A}}, \pi_{X^{\otimes n}})$ of $X^{\otimes n}$ where $\pi_{X^{\otimes n}}(x_1 \otimes \dots \otimes x_n) = \pi_X(x_1)^\epsilon \dots \pi_X(x_n)^\epsilon$, $\epsilon = 1$ when $n \geq 0$ and $\epsilon = *$ otherwise, cf. [AEE98, Lem. 2.5] or [Kwa, Thm. 3.12].

One can get a copy of $\mathcal{A} \rtimes_X \mathbb{Z}$ by a construction which generalizes regular representations for automorphism C^* -algebras. Let us briefly sketch it.

Example 1.2 (regular representation of the crossed product $\mathcal{A} \rtimes_X \mathbb{Z}$). Let π be a faithful representation of \mathcal{A} in a Hilbert space H . We define a covariant representation $(\pi_{\mathcal{A}}, \pi_X)$ of X in the Hilbert space

$$\tilde{H} := \bigoplus_{n \in \mathbb{Z}} H_n, \quad \text{where } H_n := X^{\otimes n} \otimes_{\pi} H, \quad n \in \mathbb{Z}.$$

For each n we let $\pi_{\mathcal{A}} : \mathcal{A} \rightarrow \tilde{H}$ to act on H_n as the representation induced from π by $X^{\otimes n}$. Namely, for $a \in \mathcal{A}$ we demand that $\pi_{\mathcal{A}}(a)H_n \subset H_n$ and

$$\pi_{\mathcal{A}}(a)|_{H_n} := X^{\otimes n}\text{-Ind } \pi(a), \quad n \in \mathbb{Z}.$$

For $x \in X$ we define $\pi_X(x) \in L(\tilde{H})$ such that for $n \in \mathbb{Z}$, $\pi_X(x)H_n \subset H_{n+1}$ and

$$\begin{aligned} \pi_X(x) (x_1 \otimes \dots \otimes x_n \otimes_\pi h) &= x \otimes x_1 \otimes \dots \otimes x_n \otimes_\pi h, & \text{if } n \geq 0, \\ \pi_X(x) (\flat(x_1) \otimes \dots \otimes \flat(x_{|n|}) \otimes_\pi h) &= {}_{\mathcal{A}}\langle x, x_1 \rangle \flat(x_2) \otimes \dots \otimes \flat(x_{|n|}) \otimes_\pi h, & \text{if } n < 0. \end{aligned}$$

Routine computations (making explicit use of relation (4)) show that $(\pi_{\mathcal{A}}, \pi_X)$ is indeed a covariant representation of (\mathcal{A}, X) and hence it integrates to the representation $(\pi_{\mathcal{A}} \rtimes \pi_X)$ of $\mathcal{A} \rtimes_X \mathbb{Z}$. In view of Theorem 0.1, to see that $(\pi_{\mathcal{A}} \rtimes \pi_X)$ is faithful it suffices to show that the formula

$$(10) \quad \mathcal{E} \left(\sum_{k=1}^n \pi_{\tilde{X}^{\otimes k}}(a_{-k}) + \pi_{\mathcal{A}}(a_0) + \sum_{k=1}^n \pi_{X^{\otimes k}}(a_k) \right) = \pi_{\mathcal{A}}(a_0),$$

where $a_k \in X^{\otimes k}$, $k = 0, \pm 1, \dots, \pm n$, defines a conditional expectation from the C^* -algebra $C^*(\pi_{\mathcal{A}}(\mathcal{A}), \pi_X(X))$ generated by $\pi_{\mathcal{A}}(\mathcal{A})$ and $\pi_X(X)$ onto the C^* -algebra $\pi_{\mathcal{A}}(\mathcal{A})$. The standard argument here applies, see for instance [ABL, 3.5].

A Hilbert C^* -bimodule X defines, via Rieffel's induced representation functor, a partial dynamical system on the spectrum $\hat{\mathcal{A}}$ of \mathcal{A} . More precisely, for $n \in \mathbb{Z}$ we put

$$D_n := \langle X^{\otimes n}, X^{\otimes n} \rangle_{\mathcal{A}} = \overline{\text{span}}\{\langle x, y \rangle_{\mathcal{A}} : x, y \in X^{\otimes n}\}.$$

Then by definition $D_0 = \mathcal{A}$ and $D_{-n} = {}_{\mathcal{A}}\langle X^{\otimes n}, X^{\otimes n} \rangle$. Since for each $n \in \mathbb{Z}$, D_n is an ideal in \mathcal{A} we treat its spectrum $\widehat{D_n}$ as an open set in $\hat{\mathcal{A}}$. Plainly, $X^{\otimes n}$ is a $D_{-n} - D_n$ -imprimitivity bimodule and thus the induced representation functor $X\text{-Ind}_{D_n}^{D_{-n}}$ yields a homeomorphism $\hat{h}_n : \widehat{D_n} \rightarrow \widehat{D_{-n}}$, see [RW98, Prop. 3.24, Thm. 3.29]. The system $(\{\widehat{D_n}\}_{n \in \mathbb{Z}}, \{\hat{h}_n\}_{n \in \mathbb{Z}})$ forms a partial action of \mathbb{Z} on $\hat{\mathcal{A}}$ which is generated by a single partial homeomorphism $\hat{h} : \widehat{D_1} \rightarrow \widehat{D_{-1}}$ of $\hat{\mathcal{A}}$ where

$$\hat{h} = X\text{-Ind} \quad \text{and} \quad \hat{h}^{-1} = \tilde{X}\text{-Ind}.$$

In particular, $\widehat{D_n}$ is a natural domain of h^n and $h^n = h_n$ on $\widehat{D_n}$, $n \in \mathbb{Z}$.

We adapt to non-Hausdorff spaces the standard topological freeness notion for partial homeomorphisms, see e.g. [ELQ02]. In the locally compact Hausdorff space case it reduces, by Baire theorem, to requirement that the set of periodic points has empty interior.

Definition 1.3. We say that a partial homeomorphism φ of a topological space, i.e. a homeomorphism between open subsets, is *topologically free* if for any $n \in \mathbb{N}$ and any nonempty open set U contained in a domain of φ^n there is a point $x \in U$ such that all the iterates $\varphi^k(x)$, $k = 1, 2, \dots, n$ are distinct.

The main result of the paper could be stated as follows.

Theorem 1.4 (uniqueness theorem for Hilbert bimodules). *Suppose that the partial homeomorphism $\hat{h} = X\text{-Ind}$ is topologically free. Then every covariant representation $(\pi_{\mathcal{A}}, \pi_X)$ of X integrates to the isomorphism $(\pi_{\mathcal{A}} \rtimes \pi_X)$ of $\mathcal{A} \rtimes_X \mathbb{Z}$ onto the C^* -algebra $C^*(\pi_{\mathcal{A}}(\mathcal{A}), \pi_X(X))$ generated by $\pi_{\mathcal{A}}(\mathcal{A})$ and $\pi_X(X)$ (that is the pair $(\mathcal{A} \cup X, \mathcal{R})$, cf. Definition 1.1, possess uniqueness property).*

Remark 1.5. The map \hat{h} is a lift of the partial homeomorphism $h : \text{Prim } D_1 \rightarrow \text{Prim } D_{-1}$ of $\text{Prim } \mathcal{A}$ where $h(\ker \pi) := \ker \hat{h}(\pi)$. Actually h is the restriction of

the Rieffel isomorphism between the ideal lattices of D_1 and D_{-1} given by

$$(11) \quad h(J) = \mathcal{A}\langle XJ, X \rangle, \quad h^{-1}(K) = \langle X, KX \rangle_{\mathcal{A}},$$

cf. [RW98]. Plainly, topological freeness of the system $(\text{Prim}(\mathcal{A}), h)$ implies the topological freeness of $(\widehat{\mathcal{A}}, \widehat{h})$. However, the converse is not true, see Remark 4.18 below, and thus Theorem 1.4 is not only a generalization but also a strengthening of the known isomorphism theorems for full and partial crossed-products, where topological freeness on the level of $\text{Prim}(\mathcal{A})$ was assumed, see [AL94], [Leb05].

The crossed product $\mathcal{A} \rtimes_X \mathbb{Z}$ is equipped with a semi-saturated circle gauge action γ given on generators by

$$(12) \quad \gamma_\lambda(a) = a, \quad a \in \mathcal{A}, \quad \gamma_\lambda(x) = \lambda x, \quad x \in X$$

(its zeroth and first spectral subspaces are respectively \mathcal{A} and X). Conversely, it follows from Theorem 0.1, cf. also [AEE98, Thm. 3.1], that any C^* -algebra \mathcal{B} with a semi-saturated circle action γ is naturally isomorphic to the crossed product $\mathcal{B}_0 \rtimes_{\mathcal{B}_1} \mathbb{Z}$ where \mathcal{B}_1 is the first spectral subspace for γ treated as a Hilbert bimodule over the fixed point algebra \mathcal{B}_0 . Thus one may consider Theorem 1.4 as a statement about pairs (\mathcal{B}, γ) , and the topological freeness of \widehat{h} is a property completely determined by the gauge action γ . This leads us to the following alternative version of Theorem 1.4.

Theorem 1.6. *Suppose that \mathcal{B} is equipped with a semi-saturated circle action whose first spectral subspace \mathcal{B}_1 acts topologically freely via Rieffel's induced representation functor on the spectrum of its fixed point C^* -algebra \mathcal{B}_0 . Then each homomorphism $\Psi : \mathcal{B} \rightarrow \mathcal{B}'$ which is injective on $\mathcal{B}_0 \subset \mathcal{B}$ is automatically isometric on \mathcal{B} .*

In view of Theorem 0.1 to prove Theorem 1.4 it suffices to show that (10) defines a conditional expectation for any covariant representation (π_A, π_X) . This follows immediately from Proposition 1.8 below, and among the technical instruments of the proof of this latter statement is the following simple fact, see e.g. [AL94, Lem. 12.15].

Lemma 1.7. *Let B be a C^* -subalgebra of an algebra $L(H)$. If $P_1, P_2 \in B'$ are two orthogonal projections such that the restrictions*

$$B|_{H_1} \quad \text{and} \quad B|_{H_2}$$

(where $H_1 = P_1(H)$, $H_2 = P_2(H)$) are both irreducible and these restrictions are distinct representations, then

$$H_1 \perp H_2.$$

Proposition 1.8. *Let the Rieffel homeomorphism \widehat{h} be topologically free. Assume that \mathcal{A} and X are faithfully represented in $L(H)$ so that the module actions and inner products become inherited from $L(H)$, and let b be an operator of the form*

$$(13) \quad b = \sum_{k=1}^n a_{-k}^* + a_0 + \sum_{k=1}^n a_k$$

where $a_{\pm k} \in X^{\otimes k}$, $k = 0, 1, \dots, n$ (we identify $X^{\otimes k}$ with X^k , cf. [AEE98, Lem. 2.5]). Then for every $\varepsilon > 0$ there exists an irreducible representation $\pi : \mathcal{A} \rightarrow L(H_\pi)$

such that for any irreducible representation $\nu : C^*(\mathcal{A}, X) \rightarrow L(H_\nu)$ which is an extension of π ($H_\pi \subset H_\nu$) we have

- (i) $\|\pi(a_0)\| \geq \|a_0\| - \varepsilon$,
- (ii) $P_\pi \pi(a_0) P_\pi = P_\pi \nu(b) P_\pi$,

where $P_\pi \in L(H_\nu)$ is the orthogonal projection onto H_π .

Proof. Let $\varepsilon > 0$. Since for every $a \in \mathcal{A}$ the function $\pi \rightarrow \|\pi(a)\|$ is lower semicontinuous on $\widehat{\mathcal{A}}$ and attains its upper bound equal to $\|a\|$, there exists an open set $U \subset \widehat{\mathcal{A}}$ such that

$$\|\pi(a_0)\| > \|a_0\| - \varepsilon \text{ for every } \pi \in U.$$

By topological freeness of \widehat{h} we may find $\pi \in U$ such that all the points $\widehat{h}^k(\pi)$, $k = 1, \dots, n$ are distinct (if they are defined, i.e. if $\pi(D_k) \neq 0$). Let ν be any extension of π up to an irreducible representation of $C^*(\mathcal{A}, X)$ and denote by H_π and H_ν the corresponding representation spaces for π and ν :

$$H_\pi \subset H_\nu.$$

Item (i) follows from the choice of π . To prove (ii) we need to show that for the orthogonal projection $P_\pi : H_\nu \rightarrow H_\pi$ and any element $a_k \in X^{\otimes k}$, $k \neq 0$, of the sum (13) we have

$$P_\pi \nu(a_k) P_\pi = 0.$$

We consider the case $k > 0$, the case $k < 0$ will follow by symmetry. There are two essentially different possible positions of π .

If $\pi \notin \widehat{D}_k \cap \widehat{D}_{-k}$, then either $\pi(D_k) = 0$ or $\pi(D_{-k}) = 0$. By Hewitt-Cohen Theorem (see, for example, [RW98, Prop. 2.31]) operator a_k may be presented in a form $a_k = d_- a d_+$ where $d_\pm \in D_{\pm k}$, $a \in X^{\otimes k}$, and thus

$$P_\pi \nu(a_k) P_\pi = P_\pi \nu(d_- a_k d_+) P_\pi = P_\pi \pi(d_-) P_\pi \nu(a_k) P_\pi \pi(d_+) P_\pi = 0,$$

Suppose then that $\pi \in \widehat{D}_k \cap \widehat{D}_{-k}$. Accordingly, π may be treated as an irreducible representation for both D_k and D_{-k} . We will use Lemma 1.7 where the role of P_1 is played by P_π and P_2 is the orthogonal projection onto the space

$$H_2 := \nu(X^{\otimes k}) H_\pi.$$

Clearly, $P_\pi \nu(\mathcal{A})'$ and to see that $P_2 \in \nu(\mathcal{A})'$ it suffices to note that for $a \in \mathcal{A}$, $x \in X^{\otimes k}$ and $h \in H_\pi$ we have $\nu(a)\nu(x)h = \nu(ax)h \in H_2$, that is $\nu(a)P_\pi = P_\pi \nu(a)P_\pi$, since using the same relations for $\nu(a^*)$ one gets $\nu(a)P_\pi = P_\pi \nu(a)$. Moreover, for the representation $\pi_2 : \mathcal{A} \rightarrow L(H_2)$ given by $\pi_2(a) = \nu(a)|_{H_2}$, we have $\pi_2 \cong X^{\otimes k}$ -Ind(π), or equivalently

$$\pi_2 = \widehat{h}^k(\pi).$$

Indeed, one checks that $\nu(a)h \mapsto a \otimes h$ extends to a unitary operator $V : H_2 \rightarrow X^{\otimes k} \otimes H_\pi$ that establishes the desired equivalence. Consequently, π and π_2 may be treated as irreducible representations of D_{-k} , and by the choice of π these representations are different (actually even not equivalent). Hence, by Lemma 1.7

$$P_\pi \cdot P_2 = 0$$

from which we have

$$P_\pi \nu(a_k) P_\pi = P_\pi \cdot P_2 \nu(a_k) P_\pi = 0.$$

□

1.1. Corollaries of the uniqueness theorem. One of the equivalent forms of Theorem 1.4 states that if the partial homeomorphism \widehat{h} is topologically free, then every non-trivial ideal $\mathcal{A} \rtimes_X \mathbb{Z}$ leaves an "imprint" in \mathcal{A} – has a non-trivial intersection with \mathcal{A} . By specifying these imprints one may determine the ideal structure of $\mathcal{A} \rtimes_X \mathbb{Z}$. To this end we adopt the following definition partially formulated in a general setting of partial mappings of a topological space (i.e. mappings defined on open subsets).

Definition 1.9. We say that a set V is *invariant* under a partial mapping φ with a domain Δ if

$$\varphi(V \cap \Delta) = V \cap \varphi(\Delta).$$

If there are no non-trivial closed invariant sets, then φ is called *minimal*. A partial homeomorphism φ is said to be *free*, if it is topologically free on every closed invariant set (in the case of Hausdorff space this amounts to requiring that φ has no periodic points).

Similarly to topological freeness, cf. Remark 1.5, the freeness of h is stronger condition than freeness of \widehat{h} . However, using (11) one sees that the minimality of \widehat{h} and h are equivalent, and moreover, if I is an ideal in \mathcal{A} , then the open set \widehat{I} in $\widehat{\mathcal{A}}$ is invariant if and only if

$$(14) \quad IX = XI.$$

Ideals satisfying (14) are called X -invariant in [Kat07] (X -invariant and saturated in [Kwa]). It is known, see [Kat07, 10.6] or [Kwa, Thm 7.11], that the map

$$J \mapsto J \cap \mathcal{A}$$

defines a homomorphism from the lattice of ideals in $\mathcal{A} \rtimes_X \mathbb{Z}$ onto the lattice of ideals satisfying (14). When restricted to gauge invariant ideals, i.e. ideals preserved under the gauge circle action (12), this homomorphism is actually an isomorphism. Thus if one is able to show that all ideals in $\mathcal{A} \rtimes_X \mathbb{Z}$ are gauge invariant, one obtains a complete description of the ideal structure of $\mathcal{A} \rtimes_X \mathbb{Z}$.

Theorem 1.10 (ideal lattice description). *Suppose the partial homeomorphism \widehat{h} is free. Then the map*

$$(15) \quad J \mapsto \widehat{J \cap \mathcal{A}}$$

is a lattice isomorphism between ideals in $\mathcal{A} \rtimes_X \mathbb{Z}$ and open invariant sets in $\widehat{\mathcal{A}}$. Accordingly, all ideals in $\mathcal{A} \rtimes_X \mathbb{Z}$ are gauge invariant.

Proof. It suffices to show that the map (15) is injective. To this end suppose that J is an ideal in $\mathcal{A} \rtimes_X \mathbb{Z}$, let $I = J \cap \mathcal{A}$ and denote by $\langle I \rangle$ the ideal generated by I in $\mathcal{A} \rtimes_X \mathbb{Z}$. Clearly, $\langle I \rangle \subset J$ and to prove that $\langle I \rangle = J$ we consider the homomorphism

$$\Psi : \mathcal{A} \rtimes_X \mathbb{Z} \rightarrow \mathcal{A}/I \rtimes_{X/XI} \mathbb{Z}$$

arising from the composition of the quotient maps and the universal covariant representation of $(\mathcal{A}/I, X/XI)$. Then $\ker \Psi = \langle I \rangle$ and we claim that $\Psi(J) \cap \mathcal{A}/I = \{0\}$. Indeed, if $b \in \Psi(J) \cap \mathcal{A}/I$, then $b = \Psi(a)$ for some $a \in J$ and $b = \Psi(a_1)$ for some $a_1 \in \mathcal{A}$. Thus $a - a_1 \in \ker \Psi = \langle I \rangle \subset J$ and it follows that a_1 itself is in J .

But then $a_1 \in J \cap \mathcal{A} = I$, so $b = \Psi(a_1) = 0$, which proves our claim. The system dual to $(\mathcal{A}/I, X/XI)$ naturally identifies with $(\widehat{\mathcal{A}} \setminus \widehat{I}, \widehat{h})$ and thus by freeness of $(\widehat{\mathcal{A}}, \widehat{h})$ it is topologically free. Hence Theorem 1.4 implies that $\Psi(J)$ is trivial in $\mathcal{A}/I \rtimes_{X/XI} \mathbb{Z}$. Hence $J = \langle I \rangle = \ker \Psi$. \square

Corollary 1.11 (simplicity criterion). *If the partial homeomorphism \widehat{h} is topologically free and minimal, then $\mathcal{A} \rtimes_X \mathbb{Z}$ is simple.*

2. PARTIAL CROSSED PRODUCTS AND CROSSED PRODUCTS BY INTERACTIONS. NONCOMMUTATIVE REVERSIBLE DYNAMICS

In this section we apply results obtained in the previous section to partial crossed products, crossed products by endomorphisms with complete transfer operators and crossed products by interactions. All these algebras could be considered as C^* -algebras associated with reversible noncommutative systems, and in particular are relatively easy identified with crossed products by Hilbert bimodules.

2.1. Partial crossed products. Let (θ, I, J) be a partial automorphism of a C^* -algebra \mathcal{A} , as in [Exe94, 3.1], that is I and J are ideals in \mathcal{A} and $\theta : I \rightarrow J$ is an isomorphism. A covariant representation of (θ, I, J) is a pair (π, S) where $\pi : \mathcal{A} \rightarrow L(H)$ is a representation and $S \in L(H)$ is a partial isometry such that

$$S^*SH = \pi(I)H, \quad SS^*H = \pi(J)H, \quad \text{and} \quad \pi(\theta(a)) = S\pi(a)S^*, \quad \text{for all } a \in I.$$

The partial crossed product $\mathcal{A} \rtimes_{\theta} \mathbb{Z}$ introduced in [Exe94] is a universal C^* -algebra with respect to covariant representations of (θ, I, J) . To identify $\mathcal{A} \rtimes_{\theta} \mathbb{Z}$ with a crossed product by a Hilbert bimodule we recall, cf. e.g [AEE98, Ex. 3.2], [MS98, Ex 2.22], that the space $X := J$ with actions and the inner products given by

$$a \cdot x := ax, \quad x \cdot a := \theta(\theta^{-1}(x)a), \quad \langle x, y \rangle_{\mathcal{A}} := \theta^{-1}(x^*y), \quad {}_{\mathcal{A}}\langle x, y \rangle := xy^*,$$

is a Hilbert C^* -bimodule over \mathcal{A} . Moreover, the relations

$$\pi = \pi_{\mathcal{A}}, \quad \pi_X(x) = \pi(x)S, \quad x \in J,$$

yield a one-to-one correspondence between covariant representations of (θ, I, J) and representations of (\mathcal{A}, X) . Therefore $\mathcal{A} \rtimes_{\theta} \mathbb{Z} \cong \mathcal{A} \rtimes_X \mathbb{Z}$. In the notation of the previous section one sees that $D_1 = J$, $D_{-1} = I$ and the induced representation homeomorphism $\widehat{h} : \widehat{D}_1 \rightarrow \widehat{D}_{-1}$ coincides with the inverse to $\widehat{\theta} : \widehat{I} \rightarrow \widehat{J}$ where

$$\widehat{\theta}(\pi) = \pi \circ \theta.$$

In this way, using Theorem 1.4, we arrive at the result which in the case of the group \mathbb{Z} is a strengthening (see Remark 1.5) of the main result of [Leb05].

Theorem 2.1. *If the partial homeomorphism $\widehat{\theta}$ dual to θ is topologically free, then for every faithful covariant representation (π, S) of (θ, I, J) the integrated representation $(\pi \times S)$ of $\mathcal{A} \rtimes_{\theta} \mathbb{Z}$ is automatically faithful.*

By Theorem 1.10 and Corollary 1.11 we get a noncommutative generalization of [ELQ02, Thm. 3.5] for the group \mathbb{Z} .

Theorem 2.2. *If the partial homeomorphism $\widehat{\theta}$ is free, then the map*

$$J \mapsto \widehat{J \cap \mathcal{A}}$$

is a lattice isomorphism between ideals of $\mathcal{A} \rtimes_{\theta} \mathbb{Z}$ and open $\widehat{\theta}$ -invariant sets in $\widehat{\mathcal{A}}$. In particular, if $\widehat{\theta}$ is topologically free and minimal, then $\mathcal{A} \rtimes_{\theta} \mathbb{Z}$ is simple.

2.2. Crossed products by interactions and complete C^* -dynamical systems. Throughout this subsection we fix a unital C^* -algebra \mathcal{A} . Since it is instructive to consider the notion of interaction on \mathcal{A} as a generalization of endomorphisms and transfer operators we start with recalling the latter.

A *transfer operator*, as introduced in [Exe03], for an endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ is a positive linear map $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$ which satisfies

$$(16) \quad \mathcal{L}(\alpha(a)b) = a\mathcal{L}(b), \quad a, b \in \mathcal{A};$$

then \mathcal{L} is automatically continuous, $*$ -preserving and by passing to adjoints one also gets $\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)a$, $a, b \in \mathcal{A}$. A transfer operator \mathcal{L} is said to be *non-degenerate* if $\alpha(\mathcal{L}(1)) = \alpha(1)$, and this is equivalent, cf. [Exe03, Prop. 2.3], to stating that the mapping

$$(17) \quad \mathcal{E}(a) := \alpha(\mathcal{L}(a))$$

is a conditional expectations from \mathcal{A} onto $\alpha(\mathcal{A})$. If \mathcal{L} is a non-degenerate transfer operator for (\mathcal{A}, α) , then $\mathcal{L}(1)$ is a central projection in \mathcal{A} , $1 - \mathcal{L}(1)$ is the unit in $\ker \alpha$ and $\mathcal{L}(\mathcal{A}) = \mathcal{L}(1)\mathcal{A} = (\ker \alpha)^{\perp}$, cf. [Kwa11], [BL] or [ABL].

Definition 2.3. A pair (α, \mathcal{L}) where $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$ is a non-degenerate transfer operator for an endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ shall be called a *C^* -dynamical system*.

Remark 2.4. We note that if (α, \mathcal{L}) is a C^* -dynamical system, then \mathcal{L}^n is a transfer operator for α^n for each $n \in \mathbb{N}$, however, $(\alpha^n, \mathcal{L}^n)$ may fail to be a C^* -dynamical system since \mathcal{L}^n may not be non-degenerate. In particular, this phenomena occurs in case of C^* -dynamical systems associated with graphs, cf. Remark 4.11.

A certain dissatisfaction concerning asymmetry in the definition of the pair (α, \mathcal{L}) (α is multiplicative while \mathcal{L} is “merely” positive linear) lead the author of [Exe07] to the following notion.

Definition 2.5 ([Exe07], Defn. 3.1). Let $(\mathcal{V}, \mathcal{H})$ be a pair of positive bounded linear maps $\mathcal{V}, \mathcal{H} : \mathcal{A} \rightarrow \mathcal{A}$. The pair $(\mathcal{V}, \mathcal{H})$ will be called an *interaction* if the following conditions are satisfied

- (i) $\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} = \mathcal{V}$,
- (ii) $\mathcal{H} \circ \mathcal{V} \circ \mathcal{H} = \mathcal{H}$,
- (iii) $\mathcal{V}(ab) = \mathcal{V}(a)\mathcal{V}(b)$, if either a or b belong to $\mathcal{H}(\mathcal{A})$,
- (iv) $\mathcal{H}(ab) = \mathcal{H}(a)\mathcal{H}(b)$, if either a or b belong to $\mathcal{V}(\mathcal{A})$.

If $(\mathcal{V}, \mathcal{H})$ is an interaction one shows, cf. [Exe07, Prop. 2.6, 2.7], that $\mathcal{V}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$ are C^* -subalgebras of \mathcal{A} , $\mathcal{E}_{\mathcal{V}} := \mathcal{V} \circ \mathcal{H}$ is a conditional expectation onto $\mathcal{V}(\mathcal{A})$, $\mathcal{E}_{\mathcal{H}} := \mathcal{H} \circ \mathcal{V}$ is a conditional expectation onto $\mathcal{H}(\mathcal{A})$, and the mappings

$$\mathcal{V} : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{V}(\mathcal{A}), \quad \mathcal{H} : \mathcal{V}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{A})$$

are isomorphisms, each being the inverse of the other. Since we assume that \mathcal{A} is unital we may reveal more of the structure of the pair $(\mathcal{V}, \mathcal{H})$.

Lemma 2.6. *For any interaction $(\mathcal{V}, \mathcal{H})$ the elements $\mathcal{V}(1)$ and $\mathcal{H}(1)$ are units in $\mathcal{V}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$, respectively (in particular, they are self-adjoint projections)*

Proof. Let us observe first that

$$\begin{aligned}\mathcal{E}_{\mathcal{V}}(1) &= \mathcal{V}(\mathcal{H}(1)) = \mathcal{V}(\mathcal{H}(1)1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}(1) = \mathcal{V}(\mathcal{H}(1))\mathcal{V}(\mathcal{H}(\mathcal{V}(1))) \\ &= \mathcal{V}(\mathcal{H}(1)\mathcal{H}(\mathcal{V}(1))) = \mathcal{V}(\mathcal{H}(1\mathcal{V}(1))) = \mathcal{V}(\mathcal{H}(\mathcal{V}(1))) = \mathcal{V}(1)\end{aligned}$$

and thus we have

$$\mathcal{V}(a) = \mathcal{E}_{\mathcal{V}}(\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1\mathcal{V}(a)) = \mathcal{E}_{\mathcal{V}}(1)\mathcal{V}(a) = \mathcal{V}(1)\mathcal{E}_{\mathcal{V}}(\mathcal{V}(a)1) = \mathcal{V}(1)\mathcal{V}(a)\mathcal{V}(1).$$

Hence $\mathcal{V}(1)$ is the unit in $\mathcal{V}(\mathcal{A})$ and the similar argument works for \mathcal{H} . \square

Proposition 2.7. *Any C^* -dynamical system (α, \mathcal{L}) is an interaction.*

Proof. The non-degeneracy of \mathcal{L} implies that $\alpha \circ \mathcal{L} \circ \alpha = \mathcal{E} \circ \alpha = \alpha$. Using (16) we get

$$\mathcal{L}(\alpha(\mathcal{L}(a))) = \mathcal{L}(1\alpha(\mathcal{L}(a))) = \mathcal{L}(1)\mathcal{L}(a) = \mathcal{L}(a).$$

Since α is multiplicative the condition (iii) of Definition 2.5 is trivial, and condition (iv) follow since

$$\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b = \mathcal{L}(a)\mathcal{L}(1)b = \mathcal{L}(a)\mathcal{L}(\alpha(\mathcal{L}(1)b)) = \mathcal{L}(a)\mathcal{L}(\alpha(b)),$$

and by passing to adjoints one also gets $\mathcal{L}(\alpha(b)a) = \mathcal{L}(\alpha(b))\mathcal{L}(a)$. \square

The crossed product elaborated in [ABL] (and in the semi-group setting in [KL09]), relies on an important special case of a C^* -dynamical system (α, \mathcal{L}) where the transfer operator \mathcal{L} is such that the conditional expectation (17) is given by the formula

$$\mathcal{E}(a) = \alpha(1)a\alpha(1), \quad a \in \mathcal{A}.$$

Such a transfer operator was called in [BL] a *complete transfer operator* and the corresponding system (α, \mathcal{L}) , see [Kwa11], is called *complete C^* -dynamical systems*. A complete transfer operator for a given endomorphism α exists if and only if $\ker \alpha$ is unital and $\alpha(\mathcal{A})$ is hereditary in \mathcal{A} , and then it is a unique non-degenerate transfer operator for α , cf. e.g. [Kwa11]. We naturally generalize the notion of a complete dynamical system to interactions.

Definition 2.8. An interaction $(\mathcal{V}, \mathcal{H})$ such that $\mathcal{V}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$ are hereditary subalgebras of \mathcal{A} will be called a *complete interaction*.

Proposition 2.9. *An interaction $(\mathcal{V}, \mathcal{H})$ is complete if and only if $\mathcal{V}(\mathcal{A}) = \mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$ and $\mathcal{H}(\mathcal{A}) = \mathcal{H}(1)\mathcal{A}\mathcal{H}(1)$.*

Moreover, for a complete interaction $(\mathcal{V}, \mathcal{H})$ the following conditions are equivalent

- i) $(\mathcal{V}, \mathcal{H})$ is a (necessarily complete) C^* -dynamical system
- ii) \mathcal{V} is multiplicative
- iii) $\mathcal{H}(1)$ lies in the center of \mathcal{A} .

Proof. Let $(\mathcal{V}, \mathcal{H})$ be an interaction. Since $\mathcal{V}(1)$ is the unit in $\mathcal{V}(\mathcal{A})$ we have $\mathcal{V}(\mathcal{A}) \subset \mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$. If we suppose that $\mathcal{V}(\mathcal{A})$ is hereditary, then $\mathcal{V}(\mathcal{A}) = \mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$; because for $a \in \mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$ such that $0 \leq a$ we have $a = \mathcal{V}(1)a\mathcal{V}(1) \leq \|a\|\mathcal{V}(1) \in \mathcal{V}(\mathcal{A})$ which implies $a \in \mathcal{V}(\mathcal{A})$. Plainly, the algebra $\mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$ is always a hereditary subalgebra of \mathcal{A} , and thus $\mathcal{V}(\mathcal{A}) = \mathcal{V}(1)\mathcal{A}\mathcal{V}(1)$ iff $\mathcal{V}(\mathcal{A})$ is a hereditary subalgebra of \mathcal{A} . The same argument works for \mathcal{H} and the first part of the assertion is proved.

Suppose now that $(\mathcal{V}, \mathcal{H})$ is a complete interaction. By the first part of assertion we have $\mathcal{E}_{\mathcal{H}}(a) = \mathcal{H}(1)a\mathcal{H}(1)$, $a \in \mathcal{A}$. The implication i) \Rightarrow ii) is trivial.

ii) \Rightarrow iii). Assume on the contrary that the projection $\mathcal{H}(1)$ is not a central element in \mathcal{A} . Then there exists $a \in \mathcal{A}$ such that $a\mathcal{H}(1) \neq \mathcal{H}(1)a\mathcal{H}(1)$. On one hand it follows that $\mathcal{H}(1)a^*a\mathcal{H}(1) \neq \mathcal{H}(1)a^*\mathcal{H}(1)a\mathcal{H}(1)$. On the other hand, by multiplicativity of \mathcal{V} we have

$$\begin{aligned}\mathcal{V}(\mathcal{H}(1)a^*a\mathcal{H}(1)) &= \mathcal{V}(a^*a) = \mathcal{V}(a^*)\mathcal{V}(a) = \mathcal{V}(\mathcal{H}(1)a^*\mathcal{H}(1))\mathcal{V}(\mathcal{H}(1)a\mathcal{H}(1)) \\ &= \mathcal{V}(\mathcal{H}(1)a^*\mathcal{H}(1)a\mathcal{H}(1)),\end{aligned}$$

and therefore, since \mathcal{V} is injective on $\mathcal{H}(\mathcal{A}) = \mathcal{H}(1)\mathcal{A}\mathcal{H}(1)$, we get $\mathcal{H}(1)a^*a\mathcal{H}(1) = \mathcal{H}(1)a^*\mathcal{H}(1)a\mathcal{H}(1)$ and arrive at a contradiction.

iii) \Rightarrow i). Suppose that $\mathcal{H}(1)$ is a central element in \mathcal{A} . Then \mathcal{V} is multiplicative because

$$\begin{aligned}\mathcal{V}(ab) &= \mathcal{V}(\mathcal{E}_{\mathcal{H}}(ab)) = \mathcal{V}(\mathcal{H}(1)ab\mathcal{H}(1)) = \mathcal{V}(a\mathcal{H}(1)b\mathcal{H}(1)) \\ &= \mathcal{V}(a\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(\mathcal{E}_{\mathcal{H}}(b)) = \mathcal{V}(a)\mathcal{V}(b),\end{aligned}$$

and \mathcal{H} is a transfer operator for \mathcal{V} because

$$\mathcal{H}(a\mathcal{V}(b)) = \mathcal{H}(a)\mathcal{H}(\mathcal{V}(b)) = \mathcal{H}(a)\mathcal{H}(1)b\mathcal{H}(1) = \mathcal{H}(a)b.$$

□

As in the case of C^* -dynamical systems, in a complete interaction each mapping determines uniquely the other.

Proposition 2.10. *Let $\mathcal{V} : \mathcal{A} \rightarrow \mathcal{A}$ be a positive linear map. There exists a positive map $\mathcal{H} : \mathcal{A} \rightarrow \mathcal{A}$ such that $(\mathcal{V}, \mathcal{H})$ is a complete interaction if and only if \mathcal{V} has a hereditary range, there is an orthogonal projection $P \in \mathcal{A}$ such that $\mathcal{V} : PAP \rightarrow \mathcal{V}(\mathcal{A})$ is an isomorphism and \mathcal{V} acts according to the formula*

$$\mathcal{V}(a) = \mathcal{V}(PaP)$$

If this the case, then P and \mathcal{H} are uniquely determined by \mathcal{V} and we have

$$(18) \quad \mathcal{H}(a) := \mathcal{V}^{-1}(\mathcal{V}(1)a\mathcal{V}(1))$$

where \mathcal{V}^{-1} is inverse to $\mathcal{V} : PAP \rightarrow \mathcal{V}(\mathcal{A})$.

Proof. If $(\mathcal{V}, \mathcal{H})$ is a complete interaction, then for $P = \mathcal{H}(1)$ the map $\mathcal{V} : PAP = \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{V}(\mathcal{A})$ is an isomorphism, $\mathcal{V}(a) = \mathcal{V}(\mathcal{E}_{\mathcal{H}}(a)) = \mathcal{V}(PaP)$ and since $\mathcal{E}_{\mathcal{V}}(a) = \mathcal{V}(\mathcal{H}(a)) = \mathcal{V}(1)a\mathcal{V}(1)$ it follows that $\mathcal{H}(a)$ is given by (18).

Conversely, if \mathcal{V} and P are as in the assertion and \mathcal{H} is given by (18), then one readily checks that $(\mathcal{V}, \mathcal{H})$ is a complete interaction.

What remains to show is the uniqueness of the projection P . To this end, suppose that $(\mathcal{V}, \mathcal{H}_i)$, $i = 1, 2$, are complete interactions. For the projections $P_1 = \mathcal{H}_1(1)$ and $P_2 = \mathcal{H}_2(1)$ we have

$$\mathcal{V}(P_1P_2P_1) = \mathcal{V}(P_2), \quad \mathcal{V}(P_2P_1P_2) = \mathcal{V}(P_1),$$

and as \mathcal{V} is injective on $\mathcal{H}_i(\mathcal{A}) = P_i\mathcal{A}P_i$, $i = 1, 2$, it follows that $P_1 = P_2$. □

We define the crossed product for complete interactions as a C^* -algebra universal with respect to the following covariant representations.

Definition 2.11. A *covariant representation* of a complete interaction $(\mathcal{V}, \mathcal{H})$ is the pair (π, S) consisting of a representation $\pi : \mathcal{A} \rightarrow L(H)$ and an operator $S \in L(H)$ such that

$$S\pi(a)S^* = \pi(\mathcal{V}(a)) \quad \text{and} \quad S^*\pi(a)S = \pi(\mathcal{H}(a)) \quad \text{for all } a \in \mathcal{A}$$

(then S is necessarily a partial isometry). If π is faithful we say (π, S) is faithful.

Remark 2.12. Any faithful covariant representation as defined above is a non-degenerate covariant representation in the sense of [Exe07], but the converse statement is false. In particular, the C^* -algebra B constructed in [Exe07, Thm. 6.3] is not generated by a covariant representation in our sense.

Definition 2.13. By the *crossed product of a complete interaction* $(\mathcal{V}, \mathcal{H})$, denoted by $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$, we mean the universal C^* -algebra $C^*(\mathcal{G}, \mathcal{R})$ generated by $\mathcal{G} = \mathcal{A} \cup \{S\}$ subject to \mathcal{R} consisting of all algebraic relations in \mathcal{A} and the covariance relations

$$SaS^* = \mathcal{V}(a) \text{ and } S^*aS = \mathcal{H}(a) \text{ for all } a \in \mathcal{A}.$$

We shall consider $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ equipped with a circle gauge action $\gamma = \{\gamma_\lambda\}_{\lambda \in \mathbb{T}}$ that acts on generators as follows

$$(19) \quad \gamma_\lambda(a) = a, \quad \gamma_\lambda(S) = \lambda S, \quad a \in \mathcal{A}, \quad \lambda \in \mathbb{T}.$$

Remark 2.14. If $(\mathcal{V}, \mathcal{H}) = (\alpha, \mathcal{L})$ is a C^* -dynamical system, that is if $\mathcal{H}(1)$ is a central element, then $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ coincides with the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ defined in [ABL], and for any covariant representation (π, S) operator S is necessarily a power partial isometry. If both $\mathcal{V}(1)$ and $\mathcal{H}(1)$ are central elements, then (θ, I, J) where $I = \mathcal{H}(\mathcal{A})$, $J = \mathcal{V}(\mathcal{A})$, $\theta = \mathcal{V}|_I$ is a partial automorphism of \mathcal{A} and $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ coincides with partial crossed product $\mathcal{A} \rtimes_\theta \mathbb{Z}$.

It is shown in [BL] that complete C^* -dynamical systems (α, \mathcal{L}) are precisely those C^* -dynamical systems which possess faithful covariant representations. Unlike in [BL], this result could be achieved by constructing an appropriate Hilbert bimodule. Namely, for an arbitrary endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ there is a natural structure of a C^* -correspondence over \mathcal{A} on the space $X := \alpha(1)\mathcal{A}$ given by

$$(20) \quad a \cdot x := \alpha(a)x, \quad x \cdot a := xa, \quad \text{and} \quad \langle x, y \rangle_{\mathcal{A}} := x^*y, \quad x, y \in X, \quad a \in \mathcal{A}.$$

By [Kwa11, Prop. 1.9] existence of the complete transfer operator \mathcal{L} is equivalent to existence of a left \mathcal{A} -valued inner product ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$ making X (with its predefined left action) a Hilbert bimodule. If this is the case, then

$$(21) \quad {}_{\mathcal{A}}\langle x, y \rangle = \mathcal{L}(xy^*), \quad \mathcal{L}(a) = {}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1) \rangle \quad x, y \in X, a \in \mathcal{A},$$

and there is a one-to-one correspondence between the covariant representations (π, S) of (α, \mathcal{L}) and covariant representations (π, π_X) of the Hilbert bimodule X where

$$\pi_X(x) = S^*\pi(x), \quad x \in X, \quad S := \pi_X(\alpha(1))^*.$$

In particular, $\mathcal{A} \rtimes_\alpha \mathbb{Z} \cong \mathcal{A} \rtimes_X \mathbb{Z}$ and the corresponding pair of generators and relations $(\mathcal{G}, \mathcal{R})$ is non-degenerate.

In order to construct a similar Hilbert bimodule for a general complete interaction we shall adopt, to our setting, Exel's construction of his generalized correspondence [Exe07].

We fix a complete interaction $(\mathcal{V}, \mathcal{H})$. Let $X_0 = \mathcal{A} \odot \mathcal{A}$ be the algebraic tensor product over the complexes, and let

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : X_0 \times X_0 \rightarrow \mathcal{A}, \quad {}_{\mathcal{A}}\langle \cdot, \cdot \rangle : X_0 \times X_0 \rightarrow \mathcal{A}$$

be the \mathcal{A} -valued sesqui-linear functions defined by

$$\langle a \odot b, c \odot d \rangle_{\mathcal{A}} = b^* \mathcal{H}(a^*c)d, \quad {}_{\mathcal{A}}\langle a \odot b, c \odot d \rangle = a \mathcal{V}(bd^*)c^*.$$

We consider the linear space X_0 as an \mathcal{A} - \mathcal{A} -bimodule with the natural module operations: $a \cdot (b \odot c) = ab \odot c$, $(a \odot b) \cdot c = a \odot bc$.

Proposition 2.15. *A quotient of X_0 becomes naturally a pre-Hilbert \mathcal{A} - \mathcal{A} -bimodule. More precisely*

- i) *the space X_0 with a function $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ (respectively ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$) becomes a right (respectively left) semi-inner product \mathcal{A} -module.*
- ii) *the corresponding semi-norms*

$$\|x\|_{\mathcal{A}} := \|\langle x, x \rangle_{\mathcal{A}}\|^{\frac{1}{2}} \quad \text{and} \quad {}_{\mathcal{A}}\|x\| := \|{}_{\mathcal{A}}\langle x, x \rangle\|^{\frac{1}{2}}$$

coincide on X_0 and thus the quotient space $X_0/\|\cdot\|$ obtained by modding out the vectors of length zero with respect to the seminorm $\|x\| := \|x\|_{\mathcal{A}} = {}_{\mathcal{A}}\|x\|$ is both a left and a right pre-Hilbert module over \mathcal{A} .

- iii) *denoting by $a \otimes b$ the canonical image of $a \odot b$ in the quotient space $X_0/\|\cdot\|$ we have*

$$ac \otimes b = a \otimes \mathcal{H}(c)b, \quad \text{if } c \in \mathcal{V}(\mathcal{A}), \quad a \otimes cb = a\mathcal{V}(c) \otimes b, \quad \text{if } c \in \mathcal{H}(\mathcal{A}).$$

and $a \otimes b = a\mathcal{V}(1) \otimes \mathcal{H}(1)b$, for all $a, b \in \mathcal{A}$.

- iv) *the inner-products in $X_0/\|\cdot\|$ satisfy the imprimitivity condition (4).*

Proof. i) All axioms of \mathcal{A} -valued semi-inner products for $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$, except the non-negativity, are straightforward. To show the non-negativity one may rewrite the proof of [Exe07, Pro. 5.2] (just erase the symbol $e_{\mathcal{H}}$ or put $e_{\mathcal{H}} = \mathcal{H}(1)$).

ii) Similarly, the proof of [Exe07, Pro. 5.4] yields that for $x = \sum_{i=1}^n a_i^* \odot b_i$ where $a_i, b_i \in \mathcal{A}$ we have

$$(22) \quad \|x\|_{\mathcal{A}} = \|\mathcal{H}(aa^*)^{\frac{1}{2}} \mathcal{H}(\mathcal{V}(bb^*))^{\frac{1}{2}}\| = \|\mathcal{V}(\mathcal{H}(aa^*))^{\frac{1}{2}} \mathcal{V}(bb^*)^{\frac{1}{2}}\| = {}_{\mathcal{A}}\|x\|$$

where $a = (a_1, \dots, a_n)^T$ and $b = (b_1, \dots, b_n)^T$ are viewed as column matrices.

iii) For the first part see the proof of [Exe07, Pro. 5.6]. The second part could be proved analogously. Namely, for every $x, y \in \mathcal{A}$ we have

$$\langle x \otimes y, a \otimes b \rangle_{\mathcal{A}} = y^* \mathcal{H}(x^* a) b = y^* \mathcal{H}(x^* a \mathcal{V}(1)) \mathcal{H}(1) b = \langle x \otimes y, a \mathcal{V}(1) \otimes \mathcal{H}(1) b \rangle_{\mathcal{A}}$$

which imply that $\|a \otimes b - a \mathcal{V}(1) \otimes \mathcal{H}(1) b\| = 0$.

iv) The form of condition (4) allows one to restrict to the case of simple tensors. Using iii) we have

$$\begin{aligned} a \otimes b \langle c \otimes d, e \otimes f \rangle_{\mathcal{A}} &= a \otimes bd^* \mathcal{H}(c^* e) f = a \otimes \mathcal{H}(1) bd^* \mathcal{H}(c^* e) f \\ &= a \mathcal{V}(\mathcal{H}(1) bd^* \mathcal{H}(c^* e)) \otimes f = a \mathcal{V}(\mathcal{H}(1) bd^*) \mathcal{V}(\mathcal{H}(c^* e)) \otimes f \\ &= a \mathcal{V}(bd^*) \mathcal{V}(1) c^* e \mathcal{V}(1) \otimes f = a \mathcal{V}(bd^*) c^* e \otimes f \\ &= {}_{\mathcal{A}}\langle a \otimes b, c \otimes d \rangle e \otimes f. \end{aligned}$$

□

Definition 2.16. We denote by X the Hilbert \mathcal{A} - \mathcal{A} -bimodule obtained by completion of the pre-Hilbert \mathcal{A} - \mathcal{A} -bimodule described in Proposition 2.15 and we call it a *Hilbert bimodule associated with the complete interaction $(\mathcal{V}, \mathcal{H})$.*

Remark 2.17. The Hilbert bimodule X could be obtained directly from the imprimitivity $\mathcal{K}_{\mathcal{V}}\text{-}\mathcal{K}_{\mathcal{H}}$ -bimodule \mathfrak{X} constructed by Exel in [Exe07] in the following way. By (22), X and \mathfrak{X} coincide as Banach spaces, and since

$$D_1 = \langle X, X \rangle_{\mathcal{A}} = \mathcal{A} \mathcal{H}(1) \mathcal{A}, \quad D_{-1} = {}_{\mathcal{A}}\langle X, X \rangle = \mathcal{A} \mathcal{V}(1) \mathcal{A},$$

X could be consider as an imprimitivity $\mathcal{AV}(1)\mathcal{A}\text{-}\mathcal{AH}(1)\mathcal{A}$ -bimodule. Furthermore, the mappings $\lambda_{\mathcal{V}} : \mathcal{A} \rightarrow \mathcal{K}_{\mathcal{V}}$, $\lambda_{\mathcal{H}} : \mathcal{A} \rightarrow \mathcal{K}_{\mathcal{H}}$, the author of [Exe07] used to define an $\mathcal{A}\text{-}\mathcal{A}$ -bimodule structure on \mathfrak{X} , when restricted respectively to $\mathcal{AV}(1)\mathcal{A}$ and $\mathcal{AH}(1)\mathcal{A}$ are $*$ -isomorphism. Hence we may use them to assume the identifications $\mathcal{K}_{\mathcal{V}} = \mathcal{AV}(1)\mathcal{A}$ and $\mathcal{K}_{\mathcal{H}} = \mathcal{AH}(1)\mathcal{A}$, and then the Exel's generalized correspondence and the Hilbert bimodule X coincide.

Remark 2.18. In the case $(\mathcal{V}, \mathcal{H}) = (\alpha, \mathcal{L})$ is a complete C^* -dynamical system, by Proposition 2.15 iii) we have

$$a \otimes b = a \otimes \mathcal{L}(1)b = a \otimes \mathcal{L}(\alpha(\mathcal{L}(1)b)) = a\alpha(\mathcal{L}(1)b) \otimes 1 = a\alpha(b) \otimes 1$$

and thus one may see that the mapping $X \ni a \otimes b \mapsto a\alpha(b) \in \mathcal{A}\alpha(1)$ extends to an isomorphism from the Hilbert bimodule X associated with the interaction (α, \mathcal{L}) onto the dual to Hilbert bimodule given by (20) and (21).

Now we are ready to identify the structure of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$.

Proposition 2.19. *We have a one-to-one correspondence between the covariant representations (π, S) of the interaction $(\mathcal{V}, \mathcal{H})$ and covariant representations (π, π_X) of the Hilbert bimodule X associated with $(\mathcal{V}, \mathcal{H})$, where*

$$\pi_X(a \otimes b) = \pi(a)S\pi(b), \quad x \in X, \quad S = \pi_X(1 \otimes 1).$$

In particular, the the Hilbert bimodule X could be naturally identified (as a Hilbert $\mathcal{A}\text{-}\mathcal{A}$ -bimodule) with the 1-spectral subspace of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$, and $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z} \cong \mathcal{A} \rtimes_X \mathbb{Z}$.

Proof. Let (π, S) be a covariant representation of $(\mathcal{V}, \mathcal{H})$. Since

$$\pi(a)S\pi(b)(\pi(c)S\pi(d))^* = \pi(a\mathcal{V}(bd^*)c^*) = \pi(\mathcal{A}\langle a \otimes b, c \otimes d \rangle)$$

one sees that $\pi_X(a \otimes b) := \pi(a)S\pi(b)$ is well defined on simple tensors and relations (8), (9) hold. By linearity π_X extends to the linear map defined on a dense subspace of X and the relations (8), (9) are also valid. In particular, relations (9) imply that π_X is contractive and thus extend onto the whole X .

Suppose now that (π, π_X) is a covariant representation of the Hilbert bimodule X and put $S := \pi_X(1 \otimes 1)$, then we have

$$S\pi(a)S^* = \pi_X((1 \otimes 1)a)\pi_X(1 \otimes 1)^* = \pi(\mathcal{A}\langle 1 \otimes a, 1 \otimes 1 \rangle) = \pi(\mathcal{V}(a))$$

and similarly

$$S^*\pi(a)S = \pi_X(1 \otimes 1)^*\pi_X(a(1 \otimes 1)) = \pi(\langle 1 \otimes 1, a \otimes 1 \rangle_{\mathcal{A}}) = \pi(\mathcal{H}(a)).$$

□

Proposition 2.20. *Let \mathfrak{X} be the generalized C^* -correspondence constructed from $(\mathcal{V}, \mathcal{H})$ in [Exe07, Sec. 5]. The crossed product $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ of the interaction $(\mathcal{V}, \mathcal{H})$ and a covariance algebra $C^*(\mathcal{A}, \mathfrak{X})$ of the pair $(\mathcal{A}, \mathfrak{X})$ defined in [Exe07, 7.12] are naturally isomorphic.*

Proof. In view of Remark 2.17 and remarks preceding [Exe07, 7.9] one sees that the Toeplitz algebra $\mathcal{T}(\mathcal{A}, \mathfrak{X})$ defined in [Exe07, 7.7] is a universal C^* -algebra generated by a copy of \mathcal{A} and $\mathfrak{X} = X$ subject to all $\mathcal{A}\text{-}\mathcal{A}$ bimodule relations plus the following ternary ring relation

$$(23) \quad xy^*z = x\langle y, z \rangle_{\mathcal{A}} = \mathcal{A}\langle x, y \rangle z, \quad x, y, z \in X$$

Then $C^*(\mathcal{A}, \mathfrak{X})$ is defined as the quotient $\mathcal{T}(\mathcal{A}, \mathfrak{X})/J$ where J is an ideal in $\mathcal{T}(\mathcal{A}, \mathfrak{X})$ generated by the elements $a - k$ such that $a \in (\ker \lambda)^\perp = \mathcal{AV}(1)\mathcal{A}$, $k \in X^*X$ (or resp. $a \in (\ker \rho)^\perp = \mathcal{AH}(1)\mathcal{A}$, $k \in XX^*$) and

$$ax = kx \quad (\text{or resp. } xa = xk) \quad \text{for all } x \in X.$$

Hence, in view of (23), in the algebra $C^*(\mathcal{A}, \mathfrak{X})$ we have

$$\langle x, y \rangle_{\mathcal{A}} = x^*y, \quad {}_{\mathcal{A}}\langle x, y \rangle = xy^*, \quad x, y \in X.$$

which implies that $C^*(\mathcal{A}, \mathfrak{X})$ is a universal C^* -algebra generated by a homomorphic image of \mathcal{A} and X subject to all the Hilbert bimodule relations, that is $C^*(\mathcal{A}, \mathfrak{X}) \cong \mathcal{A} \rtimes_X \mathbb{Z}$. \square

In the case of the complete interaction $(\mathcal{V}, \mathcal{H})$, \mathcal{V} and \mathcal{H} have equal rights. However, we somehow favour \mathcal{V} as it stands first from the left.

Definition 2.21. Let $(\mathcal{V}, \mathcal{H})$ be a complete interaction and let us identify the spectra of hereditary subalgebras $\mathcal{V}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$ in \mathcal{A} with open subsets of $\widehat{\mathcal{A}}$. Then the map $\widehat{\mathcal{V}} : \widehat{\mathcal{V}(\mathcal{A})} \rightarrow \widehat{\mathcal{H}(\mathcal{A})}$ dual to the $*$ -isomorphism $\mathcal{V} : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{V}(\mathcal{A})$ is a partial homeomorphism of $\widehat{\mathcal{A}}$ and we shall refer to the pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{V}})$ as to a *partial dynamical system dual to $(\mathcal{V}, \mathcal{H})$* .

Remark 2.22. Within the above identifications we have

$$\widehat{\mathcal{V}(\mathcal{A})} = \{\pi \in \widehat{\mathcal{A}} : \pi(\mathcal{V}(1)) \neq 0\}, \quad \widehat{\mathcal{H}(\mathcal{A})} = \{\pi \in \widehat{\mathcal{A}} : \pi(\mathcal{H}(1)) \neq 0\}$$

and for a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$ in the appropriate domain the formulae

$$\widehat{\mathcal{V}}(\pi)|_{\mathcal{H}(\mathcal{A})} = \pi \circ \mathcal{V} : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{B}(\pi(\mathcal{V}(1))H),$$

$$\widehat{\mathcal{H}}(\pi)|_{\mathcal{V}(\mathcal{A})} = \pi \circ \mathcal{H} : \mathcal{V}(\mathcal{A}) \rightarrow \mathcal{B}(\pi(\mathcal{H}(1))H),$$

determine the dual maps $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{H}}$ (obviously $\widehat{\mathcal{V}}^{-1} = \widehat{\mathcal{H}}$). In the case $(\mathcal{V}, \mathcal{H}) = (\alpha, \mathcal{L})$ is a complete C^* -dynamical system the partial homeomorphism $\widehat{\alpha} : \widehat{\alpha(\mathcal{A})} \rightarrow \widehat{\mathcal{L}(\mathcal{A})}$ of $\widehat{\mathcal{A}}$ dual to the isomorphism $\alpha : \mathcal{L}(\mathcal{A}) \rightarrow \alpha(\mathcal{A})$ and its inverse are given by

$$\widehat{\alpha}(\pi) = \pi \circ \alpha : \mathcal{A} \rightarrow \mathcal{B}(\pi(\alpha(1))H) \quad \text{and} \quad \widehat{\alpha}^{-1}(\pi)|_{\alpha(\mathcal{A})} = \pi \circ \mathcal{L}.$$

where $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$ is an (appropriate) irreducible representation.

Proposition 2.23. Let X be the Hilbert bimodule associated with a complete interaction $(\mathcal{V}, \mathcal{H})$ and let $\widehat{h} = X$ -Ind be the corresponding Rieffel partial homeomorphism. Then

$$\widehat{h} = \widehat{\mathcal{H}}.$$

where $\widehat{\mathcal{H}} = \widehat{\mathcal{V}}^{-1}$ and $(\widehat{\mathcal{A}}, \widehat{\mathcal{V}})$ is a partial dynamical system dual to $(\mathcal{V}, \mathcal{H})$.

Proof. Let $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$ be an irreducible representation such that $\pi(\mathcal{H}(1)) \neq 0$. The space $X \otimes_{\pi} H$ is spanned (meaning the closed linear span) by vectors $(a \otimes b) \otimes_{\pi} h$, $a, b \in \mathcal{A}$, $h \in H$. By (6) and (5) we have

$$\begin{aligned} \widehat{h}(\pi)(\mathcal{V}(1))(a \otimes b) \otimes_{\pi} h &= (\mathcal{V}(1)a \otimes b) \otimes_{\pi} h = (\mathcal{V}(1)a\mathcal{V}(1) \otimes b) \otimes_{\pi} h \\ &= (1 \otimes \mathcal{H}(a)b) \otimes_{\pi} h = (1 \otimes 1) \otimes_{\pi} \pi(\mathcal{H}(a))bh, \end{aligned}$$

and hence the space $H_0 := (\widehat{h}(\pi)(\mathcal{V}(1)))X \otimes_\pi H$ is spanned by the vectors of the form $(1 \otimes 1) \otimes_\pi h$, $h \in \pi(\mathcal{H}(1))H$. Moreover, since

$$\begin{aligned} \langle (1 \otimes 1) \otimes_\pi h_1, (1 \otimes 1) \otimes_\pi h_2 \rangle &= \langle h_1, \pi(\langle 1 \otimes 1, 1 \otimes 1 \rangle_{\mathcal{A}})h_2 \rangle = \langle h_1, \pi(\mathcal{H}(1))h_2 \rangle \\ &= \langle \pi(\mathcal{H}(1))h_1, \pi(\mathcal{H}(1))h_2 \rangle \end{aligned}$$

the mapping $(1 \otimes 1) \otimes_\pi h \mapsto \pi(\mathcal{H}(1))h$ extends to a unitary operator U from H_0 onto the space $\pi(\mathcal{H}(1))H$, and since for $a \in \mathcal{V}(\mathcal{A})$ we have

$$\widehat{h}(\pi)(a)(1 \otimes 1)_\pi \otimes h = (a \otimes 1)_\pi \otimes h = (1 \otimes \mathcal{H}(a))_\pi \otimes h = (1 \otimes 1) \otimes_\pi \pi(\mathcal{H}(a))h$$

it follows that U establishes unitary equivalence between $\widehat{h}(\pi) : \mathcal{V}(\mathcal{A}) \rightarrow \mathcal{B}(H_0)$ and $\pi \circ \mathcal{H} : \mathcal{V}(\mathcal{A}) \rightarrow \mathcal{B}(\pi(\mathcal{H}(1))H)$. Hence $\widehat{h} = \widehat{\mathcal{H}}$. \square

Combining the above results and Theorem 1.4 we get

Theorem 2.24. *Let $(\mathcal{V}, \mathcal{H})$ a complete interaction and suppose that the partial homeomorphism $\widehat{\mathcal{V}} : \widehat{\mathcal{V}(\mathcal{A})} \rightarrow \widehat{\mathcal{H}(\mathcal{A})}$ of $\widehat{\mathcal{A}}$ dual to $\mathcal{V} : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{V}(\mathcal{A})$ is topologically free. Then any C^* -algebra $C^*(\mathcal{A}, S)$ generated by a copy of \mathcal{A} and an operator S satisfying relations*

$$SaS^* = \mathcal{V}(a), \quad S^*aS = \mathcal{H}(a), \quad a \in \mathcal{A},$$

is naturally isomorphic to $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$.

In view of Theorem 1.10 and its corollary we get

Theorem 2.25. *Let $(\mathcal{V}, \mathcal{H})$ a complete interaction and $(\widehat{\mathcal{A}}, \widehat{\mathcal{V}})$ its dual partial dynamical system. The map*

$$J \mapsto \widehat{J \cap \mathcal{A}}$$

is a lattice isomorphism between gauge ideals of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ and open sets in $\widehat{\mathcal{A}}$ invariant under $\widehat{\mathcal{V}}$. Moreover, if $\widehat{\mathcal{V}}$ is free, then all the ideals in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ are gauge invariant. If $\widehat{\mathcal{V}}$ is minimal and topologically free, then $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is simple.

Remark 2.26. One may verify that an ideal I in \mathcal{A} satisfies (14) if and only if

$$(24) \quad \mathcal{V}(I) = \mathcal{V}(1)I\mathcal{V}(1).$$

Thus we have a lattice isomorphism between gauge invariant ideals of $\mathcal{A} \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ and ideals in \mathcal{A} for which (24) hold. In particular, $\widehat{\alpha}$ is minimal if and only if there are no nontrivial ideals satisfying (24).

3. COVARIANCE ALGEBRAS. FROM IRREVERSIBLE TO REVERSIBLE DYNAMICS

In this section we shall consider relations and associated C^* -algebras that arise from irreversible topological dynamics. One of the main features that makes the considered situation accessible through our scheme (2) is that not only initial but also the corresponding extended reversible system is commutative.

3.1. Covariance algebras, reversible extensions and topological freeness.

Let \mathcal{A} be a commutative C^* -algebra with unit. We identify $\mathcal{A} = C(M)$ with the algebra of continuous functions on a compact Hausdorff space M , and recall a one-to-one correspondence between endomorphisms of \mathcal{A} and partial dynamical systems on M . Namely, it is known, cf. for instance [KL08, Thm 2.2], that every endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ is of the form

$$\alpha(a) = \begin{cases} a(\varphi(x)) & , \quad x \in \Delta \\ 0 & , \quad x \notin \Delta \end{cases}, \quad a \in \mathcal{A} = C(M).$$

where $\varphi : \Delta \rightarrow M$ is a continuous mapping defined on a clopen subset $\Delta \subset M$. We refer to (M, φ) as to a *partial dynamical system* and denote by $X = \alpha(1)\mathcal{A}$ the C^* -correspondence associated to endomorphism α , see (20). There is a plenty of evidence that a natural candidate for a covariance algebra $C^*(M, \varphi)$ of (M, φ) is Katsura's C^* -algebra \mathcal{O}_X associated with the C^* -correspondence X , cf. [Kat04], [KL]. Equivalently, the algebra $C^*(M, \varphi)$ could be defined in terms of generators and relations as follows.

Definition 3.1. We let the *covariance algebra* $C^*(M, \varphi)$ of (M, φ) to be the universal C^* -algebra $C^*(\mathcal{G}, \mathcal{R})$ generated by $\mathcal{G} = \mathcal{A} \cup \{S\}$ subject to relations consisting of all algebraic relations in \mathcal{A} and the following covariance relations

$$(25) \quad \alpha(a) = SaS^*, \quad S^*Sa = aS^*S, \quad a \in \mathcal{A},$$

and

$$(26) \quad S^*Sa = a \text{ if and only if } a|_Y \equiv 0$$

where $Y = \overline{M \setminus \varphi(\Delta)}$.

The first from relations (25) imply that S is a power partial isometry. It is also known, see e.g. [KL] that \mathcal{A} embeds into $C^*(M, \varphi)$ (the pair $(\mathcal{G}, \mathcal{R})$ is non-degenerate). If $\varphi(\Delta)$ is open, then (26) amounts to say that $S^*S \in \mathcal{A}$ and $C^*(M, \varphi)$ coincides with the algebra investigated in [Kwa05]. If additionally φ is one-to-one, then φ is a partial homeomorphism and $C^*(M, \varphi)$ is the crossed product of a complete interaction for which (M, φ) is a dual system. Actually, in this case $C^*(M, \varphi)$ is both the partial crossed product and the crossed product by an endomorphism with a complete transfer operator \mathcal{L} where \mathcal{L} is the endomorphism associated with φ^{-1} . In general, relation (26) ensures that φ is a partial homeomorphism if and only if the mapping

$$\mathcal{L}(a) := S^*aS$$

invariantes the algebra \mathcal{A} , and then $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$ is a complete transfer operator for α . Moreover, one can always reduce investigation of $C^*(M, \varphi)$ to this reversible case, by passing to the bigger C^* -algebra

$$(27) \quad \mathcal{B} := \overline{\text{span}} \left(\bigcup_{n=0}^{\infty} \mathcal{L}^n(\mathcal{A}) \right).$$

This is the minimal C^* -algebra containing \mathcal{A} and preserved under the mapping $\mathcal{L}(a) = S^*aS$. Actually, see [Kwa], [KL08], \mathcal{B} is a commutative and both $\alpha(a) = SaS^*$ and $\mathcal{L}(a) = S^*aS$ are endomorphisms of \mathcal{B} . Hence the pair (α, \mathcal{L}) forms a complete interaction on \mathcal{B} whose dual partial dynamical system $(\widetilde{M}, \widetilde{\varphi})$ is described as follows.

Theorem 3.2 ([Kwa'], Thm. 2.5). *For any faithful representation of the pair $(\mathcal{G}, \mathcal{R})$ the algebra \mathcal{B} given by (27) is isomorphic to the algebra of continuous functions $C(\widetilde{M})$ on the space*

$$\widetilde{M} = \bigcup_{N=0}^{\infty} M_N \cup M_{\infty}$$

where

$$M_N = \{(x_0, x_1, \dots, x_N, 0, \dots) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, n = 1, \dots, N, x_N \in Y\}$$

and

$$M_{\infty} = \{(x_0, x_1, \dots) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, n \geq 1\},$$

are equipped with the product topology inherited from $\prod_{n \in \mathbb{N}} (M \cup \{0\})$, where $\{0\}$ is clopen. The mapping dual to $\alpha : \mathcal{B} \rightarrow \mathcal{B}$ is a partial homeomorphism $\widetilde{\varphi}$ defined on $\widetilde{\Delta} = \{(x_0, x_1, \dots) \in \widetilde{M} : x_0 \in \Delta\}$ via the formula

$$(28) \quad \widetilde{\varphi}(x_0, x_1, \dots) = (\varphi(x_0), x_0, x_1, \dots).$$

Then $\widetilde{\varphi}^{-1}(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$ defined on $\widetilde{\varphi}(\widetilde{\Delta}) = \{(x_0, x_1, \dots) \in \widetilde{M} : x_1 \neq 0\}$ is the dual map to $\mathcal{L} : \mathcal{B} \rightarrow \mathcal{B}$.

Definition 3.3. We shall call the system $(\widetilde{M}, \widetilde{\varphi})$ defined in the assertion of Theorem 3.2 a *natural reversible extension* of (M, φ) .

In the universal case the C^* -algebra $\mathcal{B} \cong C(\widetilde{M})$ is the fixed point algebra for the circle gauge action γ on $C^*(M, \varphi)$ defined on generators by (19). Moreover, it follows from Theorem 3.2 and the preceding discussion that $C^*(M, \varphi) = C^*(\widetilde{M}, \widetilde{\varphi}) = \mathcal{B} \rtimes_{\alpha, \mathcal{L}} \mathbb{Z}$ and thus applying Theorems 2.24 and 2.25 on the level of \mathcal{B} and identifying appropriate notions defined for $(\widetilde{M}, \widetilde{\varphi})$ in terms of (M, φ) one may obtain new results for $C^*(M, \varphi)$. This is the goal of the present section.

From the presented description of $(\widetilde{M}, \widetilde{\varphi})$ one deduces the following statement that generalizes [Kwa05, Thm 5.16]. Here $\Phi : \widetilde{M} \rightarrow M$ stands for the projection $\Phi(x_0, x_1, \dots) = x_0$ (this is a map dual to the inclusion $\mathcal{A} \subset \mathcal{B}$).

Proposition 3.4. *We have a one-to-one correspondence*

$$\widetilde{M} \supset \widetilde{U} \longmapsto U = \Phi(\widetilde{U}) \subset M$$

between open subsets of $\widetilde{F}_n = \{\widetilde{x} \in \widetilde{M} : \widetilde{\varphi}^n(\widetilde{x}) = \widetilde{x}\}$ and open subsets of $F_n = \{x \in M : \varphi^n(x) = x \text{ and } \varphi^{-1}(\varphi^k(x)) = \{\varphi^k(x)\}, \text{ for all } k = 1, \dots, n\}$.

Proof. If U is open subset of F_n , then $\widetilde{U} := \Phi^{-1}(U)$ is open and $U = \Phi(\widetilde{U})$ (by continuity and surjectivity of Φ). It follows immediately from the definition of F_n and \widetilde{M} that $\widetilde{U} \subset \widetilde{F}_n$. Conversely, let \widetilde{U} be an open subset of \widetilde{F}_n and take $\widetilde{x} = (x_0, x_1, \dots) \in \widetilde{U}$. Then $\varphi^n(x_0) = x_0$ and $x_{kn} = x_0$, for all $k \in \mathbb{N}$. Thus, by the definition of product topology there is an open neighbourhood U_{x_0} of x_0 and $k \in \mathbb{N}$ such that $\{\widetilde{y} = (y_0, y_1, \dots) \in \widetilde{M} : y_{kn} \in U_{x_0}\} \subset \widetilde{U}$. It follows that $U_{x_0} \subset F_n$ and $U_{x_0} \subset \Phi(\widetilde{U})$. As a consequence $U := \Phi(\widetilde{U})$ is an open subset of F_n . \square

In view of the above it is natural to adopt the following definition which is consistent with Definition 1.3.

Definition 3.5 ([Kwa05] Defn. 5.12). We say that the orbit of φ -periodic point x with period n has no entrance, if $\varphi^{-1}(\varphi^k(x)) = \{\varphi^k(x)\}$, for all $k = 1, \dots, n$. The partial mapping φ is *topologically free* if the set of all periodic points whose orbits have no entrance has empty interior.

By Proposition 3.4 the partial mapping φ is topologically free if and only if the partial homeomorphism $\tilde{\varphi}$ is topologically free. Therefore by Theorem 2.24 we get a uniqueness theorem for covariance algebras that generalizes [Kwa05, Thm. 6.11].

Theorem 3.6. *If φ is topologically free, then any C^* -algebra $C^*(\mathcal{A}, S)$ generated by a C^* -algebra \mathcal{A} isomorphic to $C(M)$ and an operator S satisfying relations (25), (26) is naturally isomorphic to $C^*(M, \varphi)$.*

3.2. Relative covariance algebras and their ideal structure. It is often useful, see e.g. [Kwa'], and when investigating ideal structure of $C^*(M, \varphi)$ is almost indispensable to consider a relative version $C^*(M, \varphi; Y)$ of a covariance algebra of (M, φ) where $Y \subset M$ is arbitrary closed set (not necessarily equal to $\overline{M \setminus \varphi(\Delta)}$).

Definition 3.7. We denote by $C^*(M, \varphi; Y)$ the universal C^* -algebra $C^*(\mathcal{G}, \mathcal{R})$ where $(\mathcal{G}, \mathcal{R})$ is as in the Definition 3.1 (with the exception that we do not require that $Y = \overline{M \setminus \varphi(\Delta)}$) and call it a *covariance algebra relative to Y* .

Algebra $C^*(M, \varphi; Y)$ could be considered as a special case of relative Cuntz-Pimsner algebra $\mathcal{O}(X, J)$ of P. Muhly and B. Solel [MS98] where $X = \alpha(1)\mathcal{A}$ is the C^* -correspondence associated with endomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ and $J = \{a \in \mathcal{A} = C(M) : a|_Y \equiv 0\}$. In particular, using [MS98, Prop. 2.21] one may see that \mathcal{A} embeds into $C^*(M, \varphi; Y)$ if and only if $Y \cup \varphi(\Delta) = M$. Moreover, by [Kwa', Thm. 2.5], if $Y \cup \varphi(\Delta) = M$, Theorem 3.2 remains valid and hence it seems reasonable to generalize the definition 3.3.

Definition 3.8. For an arbitrary closed set Y such that $Y \cup \varphi(\Delta) = M$ the system $(\widetilde{M}, \tilde{\varphi})$ described in the assertion of Theorem 3.2 shall be called a *natural reversible Y -extension* of (M, φ) .

In the sequel we assume that Y is closed such that $Y \cup \varphi(\Delta) = M$ and we let $(\widetilde{M}, \tilde{\varphi})$ to be the natural reversible Y -extension $(\widetilde{M}, \tilde{\varphi})$. One readily sees that, under these assumptions, the assertion of Proposition 3.4 remains true if one replaces F_n with $F_n \setminus Y$. Consequently, one could generalize topological freeness and Theorem 3.6 as follows.

Definition 3.9. Let Y be a closed set such that $Y \cup \varphi(\Delta) = M$. The partial mapping φ is said to be *topologically free outside Y* if the set of periodic points whose orbits do not intersect Y and have no entrance has empty interior.

Theorem 3.10. *Suppose that Y is topologically free outside Y . Then any C^* -algebra $C^*(\mathcal{A}, S)$ generated by a C^* -algebra \mathcal{A} isomorphic to $C(M)$ and an operator S satisfying relations (25), (26) is naturally isomorphic to $C^*(M, \varphi; Y)$.*

In order to determine the lattice structure of covariance algebras $C^*(M, \varphi; Y)$ we need to identify invariant subsets of \widetilde{M} . We start with the well-behaved special case studied in [Kwa05].

Proposition 3.11. *If $Y = M \setminus \varphi(\Delta)$ (in particular $\varphi(\Delta)$ must be open), the map*

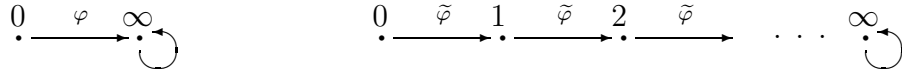
$$(29) \quad \widetilde{M} \supset \widetilde{V} \mapsto V = \Phi(\widetilde{V}) \subset M$$

is a lattice isomorphism between the lattices of the sets invariant under $\widetilde{\varphi}$ and φ , respectively (see Definition 1.9). Its inverse is given by $M \supset V \mapsto \widetilde{V} = (V \times V \cup \{0\} \times \dots) \cap \widetilde{M}$.

Proof. Suppose that \widetilde{V} is $\widetilde{\varphi}$ -invariant and note that since $\widetilde{\varphi} : \widetilde{\Delta} \rightarrow \widetilde{\varphi}(\widetilde{\Delta})$ is a bijection, \widetilde{V} is also $\widetilde{\varphi}^{-1}$ -invariant. We put $V := \Phi(\widetilde{V})$. Then $\varphi(V \cap \Delta) = \varphi(\Phi(\widetilde{V} \cap \widetilde{\Delta})) = \Phi(\widetilde{\varphi}(\widetilde{V} \cap \widetilde{\Delta})) = \Phi(\widetilde{V} \cap \widetilde{\varphi}(\widetilde{\Delta})) \subset V \cap \varphi(\Delta)$. To see the opposite inclusion let $x_0 \in V \cap \varphi(\Delta)$. Then there is $\widetilde{x} \in \widetilde{V}$ such that $\Phi(\widetilde{x}) = x_0$ and since $x_0 \notin Y = M \setminus \varphi(\Delta)$ we have $\widetilde{x} = (x_0, x_1, \dots)$ where $x_1 \neq 0$. Plainly, $\varphi(x_1) = x_0$ and $x_1 \in V$ by $\widetilde{\varphi}^{-1}$ -invariance of \widetilde{V} . Thus $x_0 \in \varphi(V \cap \Delta)$ and V is φ -invariant.

Conversely, let V be φ -invariant and let $\widetilde{V} := (V \times V \cup \{0\} \times \dots) \cap \widetilde{M}$. The inclusion $\Phi(\widetilde{V}) \subset V$ is clear, and to show the opposite inclusion we let $x_0 \in V$. One may find $\widetilde{x} \in \widetilde{V}$ such that $\widetilde{\varphi}(\widetilde{x}) = x_0$ in the following way: if $x_0 \notin \varphi(\Delta)$, that is $x_0 \in Y = M \setminus \varphi(\Delta)$, then we may put $\widetilde{x} := (x_0, 0, \dots) \in \widetilde{V}$, otherwise (by invariance of V) there is $x_1 \in V \cap \Delta$ such that $\varphi(x_1) = x_0$ and one may apply the foregoing procedure to x_1 . Proceeding in this way, one may end up with a “finite” sequence $\widetilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) \in \widetilde{V} \cap M_N$ or there is “infinite” sequence $\widetilde{x} = (x_0, x_1, x_2, \dots) \in \widetilde{V} \cap M_\infty$. Once we proved that $\Phi(\widetilde{V}) = V$ the invariance of \widetilde{V} is straightforward. \square

Remark 3.12. Plainly, (29) yields a one-to-one correspondence between invariant closed sets, however, it may fail to be a bijection between open sets. Indeed, if $M = \{0, \infty\}$, $\varphi(0) = \varphi(\infty) = \infty$ and $Y = \{0\}$, then $\widetilde{M} = \overline{\mathbb{N}}$ and $\widetilde{\varphi}(n) = n + 1$, $n \in \overline{\mathbb{N}}$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the one point compactification of the discrete space \mathbb{N} . The graphs of the corresponding dynamics are as follows



and all the φ -invariant sets $\{M, \emptyset, \{\infty\}\}$ are clopen whereas the $\widetilde{\varphi}$ -invariant sets $\{\widetilde{M}, \emptyset, \{\infty\}\}$ are all closed but $\{\infty\}$ is not open in $\widetilde{M} = \overline{\mathbb{N}}$.

Thus we see that in order to obtain ideal lattice description of $C^*(M, \varphi)$ in terms of invariant subsets of M we have to use closed sets not open ones.

Remark 3.13. We note that if the system (M, φ) is minimal two cases are possible:

- i) $\varphi : M \rightarrow M$ is a full minimal surjection;
- ii) $\varphi(x_i) = x_{i+1}$, $i = 1, \dots, n-1$ where $M = \{x_1, \dots, x_n\}$ and $\Delta = M \setminus \{x_n\}$.

In this case $C^*(M, \varphi) \cong M_n(\mathbb{C})$ is the algebra of complex $n \times n$ matrices.

The system (M, φ) is minimal and not topologically free if and only if M is finite and consists of a periodic orbit of φ . Then $C^*(M, \varphi) \cong C(\mathbb{T}, M_n(\mathbb{C}))$ is not simple.

In view of Proposition 3.11, Remark 3.13, and Theorem 2.25, we get

Theorem 3.14. *If $\varphi(\Delta)$ is open, then the map*

$$J \mapsto M \setminus \widehat{J \cap \mathcal{A}}$$

is a lattice anti-isomorphism between gauge invariant ideals in $C^*(M, \varphi)$ and closed sets invariant under φ . Moreover,

- i) if φ has no periodic points, all ideals in $C^*(M, \varphi)$ are gauge invariant;
- ii) $C^*(M, \varphi)$ is simple if and only if (M, φ) is minimal and M is not a periodic orbit.

If $Y \neq M \setminus \varphi(\Delta)$, the map (29) is neither injective nor does map invariant sets onto invariant sets. Thus a generalization of Theorem 3.14 requires a nontrivial reformulation.

Example 3.15. Let $M = [0, 1]$, $\varphi(x) = x/2$ and $Y = \overline{M \setminus \varphi(\Delta)} = [1/2, 1]$. Then the reversible Y -extension $(\widetilde{M}, \widetilde{\varphi})$ of (M, φ) could be described as follows. We identify \widetilde{M} with a subspace of $\overline{\mathbb{N}} \times [0, 1]$ by adopting the notation

$$\widetilde{M} = \left(\bigcup_{n \in \mathbb{N}} \{n\} \times \varphi^n(Y) \right) \cup \{(\infty, 0)\}, \quad \widetilde{\varphi}(n, x) = (n+1, x/2), \quad \Phi(n, x) = x,$$

where $n \in \overline{\mathbb{N}}$. One sees that every nonempty φ -invariant closed subsets of M is of the form

$$V_K = \bigcup_{n=0}^{\infty} \varphi^n(K) \cup \{0\}$$

where K is a closed subset of Y such that $1/2 \in K$ iff $1 \in K$. Hence φ -invariant sets are parametrized by subsets of a circle obtained by identification of the endpoints of $Y = [1/2, 1]$. On the other hand, all nonempty $\widetilde{\varphi}$ -invariant closed sets are of the form

$$\widetilde{V}_K = \left(\bigcup_{n \in \mathbb{N}} \{n\} \times \varphi^n(K) \right) \cup \{(\infty, 0)\}$$

where K is an arbitrary closed subset of the interval Y . In particular, if $1/2 \in K$ and $1 \notin K$, the set $\Phi(\widetilde{V}_K) = V_K$ is not invariant even though \widetilde{V}_K is. Moreover, if $1 \in K$ and $1/2 \notin K$, then \widetilde{V}_K , $\widetilde{V}_K \cup \{1/2\}$ and $\Phi(\widetilde{V}_K) = \Phi(\widetilde{V}_K \cup \{1/2\}) = V_K \cup \{1/2\}$ are invariant but $\widetilde{V}_K \neq \widetilde{V}_K \cup \{1/2\}$.

Definition 3.16. We shall say that $V \subset M$ is *positive invariant* under φ if $\varphi(V \cap \Delta) \subset V$, and V is *Y -negative invariant* if $V \subset Y \cup \varphi(V \cap \Delta)$. If V is both positive and Y -negative invariant, we shall call it *Y -invariant*.

Plainly, invariance imply Y -invariance and in the case $Y = M \setminus \varphi(\Delta)$ these notions coincide. In general, (29) maps invariant sets onto Y -invariant sets. However, as Example 3.15 shows, this mapping is not injective and we need "more data".

Definition 3.17. A pair (V, V') of closed subsets of M shall be called a *Y -pair* for the partial dynamical system (M, φ) if

$$V \text{ is positive } \varphi\text{-invariant, } V' \subset Y \text{ and } V' \cup \varphi(V \cap \Delta) = V.$$

Remark 3.18. There is a natural lattice structure on the family of Y -pairs given by the partial order: $(V_1, V'_1) \subset (V_2, V'_2) \iff V_1 \subset V_2 \text{ and } V'_1 \subset V'_2$. This lattice is bounded with the greatest element (M, Y) and the least element (\emptyset, \emptyset) .

The first component of a Y -pair (V, V') is a Y -invariant set, and in the case $Y = M \setminus \varphi(\Delta)$ the second component is determined by the first one: we necessarily have $V' = V \cap Y$. In general, a Y -pair (V, V') could be considered as a subsystem (V, φ) of (M, φ) equipped with the set V' playing the same role as Y for (M, φ) . Consequently, natural reversible V' -extension $(\tilde{V}, \tilde{\varphi})$ can be treated as a subsystem of $(\tilde{M}, \tilde{\varphi})$. Namely, $\tilde{V} = \bigcup_{N=0}^{\infty} V_N \cup V_{\infty} \subset \tilde{M}$, where $V_N = \{(x_0, x_1, \dots, x_N, 0, \dots) : x_n \in V \cap \Delta, \varphi(x_n) = x_{n-1}, n = 1, \dots, N, x_N \in V'\}$ and $V_{\infty} = \{(x_0, x_1, \dots) : x_n \in V \cap \Delta, \varphi(x_n) = x_{n-1}, n \geq 1\}$, is invariant under $\tilde{\varphi}$ given by (28).

Proposition 3.19. *The map*

$$(30) \quad \tilde{M} \supset \tilde{V} \longmapsto (V, V') := (\Phi(\tilde{V}), \Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta})))$$

is a lattice isomorphism between the lattices of the closed sets invariant under $\tilde{\varphi}$ and Y -pairs for (M, φ) . Its inverse is given by a natural reversible V' -extension of (V, φ) .

Proof. Let \tilde{V} be invariant under $\tilde{\varphi}$. The positive invariance of $V := \Phi(\tilde{V})$ under φ and the inclusion $V' := \Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta})) \subset Y$ are clear. Moreover,

$$\begin{aligned} V &= \Phi(\tilde{V} \cap \tilde{\varphi}(\tilde{\Delta}) \cup \tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta})) = \Phi(\tilde{V} \cap \tilde{\varphi}(\tilde{\Delta})) \cup \Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta})) \\ &= \Phi(\tilde{\varphi}(\tilde{V} \cap \tilde{\Delta})) \cup V' = \varphi(V \cap \Delta) \cup V'. \end{aligned}$$

Hence (V, V') is a Y -pair. Conversely, let (V, V') be an arbitrary Y -pair and let \tilde{V} be the base space of the natural reversible V' -extension of (V, φ) . The equality $\Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta})) = V'$ and inclusion $\Phi(\tilde{V}) \subset V$ are straightforward. The inclusion $\Phi(\tilde{V}) \supset V$ can be obtained similarly like in proof of Proposition 3.11, using equality $V' \cup \varphi(V \cap \Delta) = V$. The invariance of \tilde{V} under $\tilde{\varphi}$ follows from invariance of V . \square

In connection with the forthcoming statement, we note that treating $\mathcal{B} = C(\tilde{M})$ in the operator algebraic form (27) the element $\mathcal{L}(1)$ is the characteristic function of $\tilde{\varphi}(\tilde{\Delta})$.

Theorem 3.20. *If Y is closed and such that $Y \cup \varphi(\Delta) = M$, then the map*

$$J \longmapsto (M \setminus \hat{I}, M \setminus \hat{I}') \text{ where } I = J \cap \mathcal{A} \text{ and } I' = \{a \in \mathcal{A} : (1 - \mathcal{L}(1))a \in J\}$$

is a lattice anti-isomorphism between the gauge invariant ideals in $C^(M, \varphi; Y)$ and Y -pairs for φ . Moreover,*

- i) *if φ has no periodic points, all ideals in $C^*(M, \varphi; Y)$ are gauge invariant;*
- ii) *$C^*(M, \varphi; Y)$ is not simple unless $Y = M \setminus \varphi(\Delta)$, and then Theorem 3.14 apply.*

Proof. We know that gauge invariant ideals J in $C^*(M, \varphi; Y)$ correspond to sets $\tilde{V} = \tilde{M} \setminus \widehat{J \cap \mathcal{B}}$ invariant under $\tilde{\varphi}$, which in turn correspond to Y -pairs (V, V') where $V = \Phi(\tilde{V}) = M \setminus \widehat{J \cap \mathcal{A}}$ and $V' = \Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta}))$. Thus to prove the first part of assertion it suffices to describe V' in terms of J . To this end, let $a \in \mathcal{A}$ and note that

$$(1 - \mathcal{L}(1))a \in J \iff \chi_{\tilde{M} \setminus \tilde{\varphi}(\tilde{\Delta})}(a \circ \Phi) \in J \cap \mathcal{B} \iff a|_{\Phi(\tilde{V} \setminus \tilde{\varphi}(\tilde{\Delta}))} \equiv 0.$$

Item i) is clear by Theorems 3.2, 2.25. To show ii) suppose on the contrary that $C^*(M, \varphi; Y)$ is simple and $Y \neq M \setminus \varphi(\Delta)$. By simplicity of $C^*(M, \varphi; Y)$ there are

no non-trivial Y -pairs for (M, φ) , and in particular (M, φ) is minimal. However, regardless of the form of (M, φ) described in Remark 3.13, one sees that $(M, M \setminus \varphi(\Delta))$ is a non-trivial Y -pair and we arrive at a contradiction. \square

Remark 3.21. If $Y = \overline{M \setminus \varphi(\Delta)}$ the Y -pairs of sets correspond to O -pairs of ideals for the C^* -correspondence $X = \alpha(1)\mathcal{A}$ – a notion introduced in [Kat07]. In general, Y -pairs correspond to T -pairs coisometric on the ideal of functions from $\mathcal{A} = C(M)$ that vanish on Y , see [Kwa, 7.15]. In particular, one could obtain first parts of assertions of Theorems 3.14, 3.20 using [Kat07, Prop. 11.9] or [Kwa, 7.16].

Remark 3.22. In the case $Y \cup \varphi(\Delta) \neq M$ the algebra $\mathcal{A} = C(M)$ does not embed into $C^*(M, \varphi; Y)$ but there is a Y -invariant closed set R such that $C^*(M, \varphi; Y) \cong C^*(R, \varphi; Y \cap R)$; namely $R = \bigcap_{n=1}^{\infty} (\bigcup_{k=0}^{n-1} \varphi^k(Y \cap \Delta_k) \cup \varphi^n(\Delta_n))$ where Δ_n is the natural domain of φ^n , cf. [KL] or [Kwa, Ex. 6.22] (note that $R = M$ if and only if $Y \cup \varphi(\Delta) = M$). Accordingly, one may apply Theorems 3.10, 3.20 to the reduced system (R, φ) to obtain results for $C^*(M, \varphi; Y)$.

4. CUNTZ-KRIEGER ALGEBRAS. FROM IRREVERSIBLE DYNAMICS TO INTERACTIONS

In this section we show that for a Hilbert C^* -bimodule determined by the gauge action in a Cuntz-Krieger algebra the classical Cuntz-Krieger uniqueness theorem and Theorem 1.4 are equivalent. For this purpose we identify Cuntz-Krieger algebras as crossed products by interaction and describe in detail the corresponding dual reversible dynamical systems. This approach is interesting in its own right.

4.1. Cuntz-Krieger algebra of a finite graphs and its core C^* -algebra \mathcal{F}_E .

Throughout we let $E = (E^0, E^1, r, s)$ be a finite directed graph, that is E^0 is a set of vertices, E^1 is a set of edges, $|E^0|, |E^1| < \infty$, and $r, s : E^1 \rightarrow E^0$ are range, source maps. We briefly recall the related objects and notation, cf. [BPRS00], [KPR98], [KPRR97].

A *Cuntz-Krieger E -family* compose of (non-zero) orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying relations

$$(31) \quad S_e^* S_e = P_{r(e)} \quad \text{and} \quad P_v = \sum_{e \in s^{-1}(v)} S_e S_e^* \quad \text{for all } v \in s(E^1), e \in E^1.$$

The *graph C^* -algebra* or *Cuntz-Krieger algebra* $C^*(E)$ of E is a universal C^* -algebra generated by a universal Cuntz-Krieger E -family, that is $C^*(E) = C^*(\mathcal{G}, \mathcal{R})$ where $\mathcal{G} = \{P_v : v \in E^0\} \cup \{S_e : e \in E^1\}$ and \mathcal{R} consists of relations (31) plus relations: $S_e S_e^* S_e$, $e \in E^1$, and $P_v^2 = P_v^* = P_v$, $v \in E^0$. The C^* -algebra $C^*(E)$ is equipped with the natural *circle gauge action* $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(E)$ established by relations

$$(32) \quad \gamma_\lambda(P_v) = P_v, \quad \gamma_\lambda(S_e) = \lambda S_e, \quad v \in E^0, e \in E^1, \lambda \in \mathbb{T}.$$

We write E^n , $n > 0$, for the set of paths $\mu = (\mu_1, \dots, \mu_n)$, $r(e_i) = s(e_{i+1})$, $i = 1, \dots, n-1$. The maps r, s naturally extend to E^n , so that (E^0, E^n, s, r) is the graph (n -times composition of E), and s extends to the set E^∞ of infinite paths $\mu = (\mu_1, \mu_2, \dots)$. By a sink (respectively a source) we mean a vertex which does not emit (receive) any edges. In particular, we denote by $E_{\text{sinks}}^0 := E^0 \setminus s(E^1)$ the set of all sinks and $E^0 \setminus r(E^1)$ is the set of all sources in E .

If $\{P_v : v \in E^0\}$, $\{S_e : e \in E^1\}$ is a Cuntz-Krieger E -family and $\mu \in E^n$ we write $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$. We also put $S_v := P_v$ and $s(v) = r(v) = v$ for $v \in E^0$. Using the convention that for $\mu \in E^n$, $\nu \in E^m$ such that $r(\mu) = s(\nu)$, $\mu\nu = (\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_m)$ is the path in E^{m+n} , the Cuntz-Krieger E -family relations extends onto operators S_μ , see [KPR98, Lem 1.1], as follows

$$(33) \quad S_\nu^* S_\mu = \begin{cases} S_{\mu'}, & \text{if } \mu = \nu\mu', \mu' \notin E^0, \\ P_{r(\mu)} & \text{if } \mu = \nu, \\ S_{\nu'}^* & \text{if } \nu = \mu\nu', \nu' \notin E^0, \\ 0 & \text{otherwise} \end{cases}$$

It is well known that the C^* -algebra

$$(34) \quad \mathcal{F}_E = \overline{\text{span}} \{S_\mu S_\nu^* : |\mu| = |\nu| = n, n = 0, 1, \dots\}$$

(up to natural isomorphism) does not depend on the choice of the Cuntz-Krieger E -family and in the universal case it is the fixed-point algebra for the gauge action (32). We shall refer to \mathcal{F}_E as to *core C^* -algebra* when we view \mathcal{F}_E independently on its realization via (34). Actually, since we assume that E is finite, \mathcal{F}_E is an AF-algebra. We recall a standard Bratteli diagram for \mathcal{F}_E , cf. [BPRS00]. For each vertex v and $N \in \mathbb{N}$ we set

$$(35) \quad \mathcal{F}_N(v) := \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^N, r(\mu) = r(\nu) = v\}$$

which is a simple factor of type I_n with $n = |\{\mu \in E^N : r(\mu) = v\}|$ (If $n = 0$ we put $\mathcal{F}_N(v) := \{0\}$). The spaces

$$(36) \quad \mathcal{F}_N := \left(\bigoplus_{v \notin E_{\text{sinks}}^0} \mathcal{F}_N(v) \right) \oplus \left(\bigoplus_{w \in E_{\text{sinks}}^0} \bigoplus_{i=0}^N \mathcal{F}_i(w) \right), \quad N \in \mathbb{N},$$

form an increasing family of finite-dimensional algebras and

$$\mathcal{F}_E = \overline{\bigcup_{N \in \mathbb{N}} \mathcal{F}_N}.$$

We denote by $\Lambda(E)$ the corresponding Bratteli diagram for \mathcal{F}_E . If E has no sinks $\Lambda(E)$ can be obtained by an infinite vertical concatenation of E treated as bipartite graph, see [MRS92], otherwise one has to add to every sink an infinite tail, see [BPRS00]. We adopt the convention that if V is a subset of E^0 we treat it as a full subgraph of E and $\Lambda(V)$ stands for the corresponding Bratteli diagram for \mathcal{F}_V . In particular, if V is *hereditary*, that is if $s(e) \in V \implies r(e) \in V$ for all $e \in E^1$, then \mathcal{F}_V can be naturally considered as a subalgebra of \mathcal{F}_E . If additionally V is *saturated*, i.e. every vertex which feeds into V and only V is in V , then \mathcal{F}_V is an ideal in \mathcal{F}_E . Roughly speaking, viewing $\Lambda(E)$ as an infinite directed graph the hereditary and saturated subgraphs (subdiagrams) correspond to ideals in \mathcal{F}_E , see [Bra72, 3.3].

4.2. Cuntz-Krieger algebra $C^*(E)$ as crossed product of interactions. The gauge action (32) is semi-saturated and we will show that it can be naturally considered as the gauge action in a crossed product of interactions. This approach, inspired by [ABL], see also [HR], will allow us to apply Theorem 1.4 to $C^*(E)$ in a very natural and transparent way.

Let us fix a Cuntz-Krieger E -family $\{P_v : v \in E_0\}$, $\{S_e : e \in E_1\}$. For each vertex $v \in E^0$ we let

$$n_v := |r^{-1}(v)|$$

be the number of edges that v receives. Let us consider the operator

$$(37) \quad S := \sum_{e \in E_1} \frac{1}{\sqrt{n_{r(e)}}} S_e.$$

Using (33) one sees that S is a partial isometry with the initial projection $S^*S = \sum_{v \in r(E^1)} P_v$ (S is an isometry iff E has no sources). We associate with S two mappings defined on the C^* -algebra $C^*(\{S_\mu : \mu \in E^1 \cup E^0\})$:

$$(38) \quad \mathcal{V}(a) := SaS^*, \quad \mathcal{H}(a) := S^*aS.$$

Routine computations using standard relations [KPR98, Lem 1.1] show that the actions of \mathcal{V} and \mathcal{H} on \mathcal{F}_E are determined by the following formulas

$$(39) \quad \mathcal{V}(S_\mu S_\nu^*) = \begin{cases} \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}} \sum_{e,f \in E^1} S_{e\mu} S_{f\nu}^*, & n_{s(\mu)}n_{s(\nu)} \neq 0, \\ 0, & n_{s(\mu)}n_{s(\nu)} = 0, \end{cases}$$

$$(40) \quad \mathcal{H}(S_{e\mu} S_{f\nu}^*) = \frac{1}{\sqrt{n_{s(\mu)}n_{s(\nu)}}} S_\mu S_\nu^*, \quad \mathcal{H}(P_v) = \begin{cases} \sum_{e \in s^{-1}(v)} \frac{P_{r(e)}}{n_{r(e)}}, & v \notin E_{sinks}^0, \\ 0, & v \in E_{sinks}^0, \end{cases}$$

where $\mu, \nu \in E^n$, $n \in \mathbb{N}$, $e, f \in E^1$, $v \in E^0$. Plainly, $(\mathcal{V}, \mathcal{H})$ forms a complete interaction on \mathcal{F}_E and since \mathcal{F}_E does not depend on the choice of the Cuntz-Krieger family $\{P_v : v \in E_0\}$, $\{S_e : e \in E_1\}$, the pair $(\mathcal{V}, \mathcal{H})$ is uniquely determined by the graph E .

Definition 4.1. We say that the pair $(\mathcal{V}, \mathcal{H})$ where \mathcal{V} and \mathcal{H} satisfy (39), (40) is a *(complete) interaction on \mathcal{F}_E associated to the graph E* .

Unlike in the situation of the previous section, where we associated a complete interaction $(\mathcal{V}, \mathcal{H}) = (\alpha, \mathcal{L})$ to an endomorphism $\mathcal{V} = \alpha$, we would like to think that in the pair $(\mathcal{V}, \mathcal{H})$ associated to the graph E the predominant role plays \mathcal{H} not \mathcal{V} . Let us for instance note that \mathcal{H} preserves the “vertex C^* -algebra”

$$\mathcal{A}^0 := \text{span}(\{P_v : v \in E^0\}) \cong C(E^0) = \mathbb{C}^{|E^0|}$$

and acts on it as a transfer operator associated to a stochastic Markov chain. Namely, $\mathcal{H}(P_v) = \sum_{w \in E^0} p_{v,w} P_w$, $v \in E^0$, where $P = [p_{v,w}]$ is a quasi-left stochastic matrix arising from the adjacency matrix $A = [A(v, w)]_{v,w \in E^0}$ of the graph E , that is

$$(41) \quad p_{v,w} := \begin{cases} \frac{A(v,w)}{n_w}, & A(v, w) \neq 0, \\ 0, & A(v, w) = 0, \end{cases}$$

where $A(v, w) = |\{e \in E^1 : s(e) = v, r(e) = w\}|$. By a term *left quasi-stochastic matrix* we mean that each non-zero column of P sums up to one (the zero columns correspond to sources). However, unless E has no multiedges, i.e. unless $A(v, w) \in \{0, 1\}$, the system $(\mathcal{A}^0, \mathcal{H})$ does not contain satisfactory information on E ; the matrix A can not be reconstructed from $(\mathcal{A}^0, \mathcal{H})$. Therefore it might be more

natural to consider a bigger algebra as a starting object. For instance, for the “edge C^* -algebra”

$$\mathcal{A} := \text{span}(\{P_v : v \in E_{\text{sinks}}^0\} \cup \{S_e S_e^* : e \in E^1\}) \cong \mathbb{C}^{|E_{\text{sinks}}^0| + |E^1|}$$

we have $\mathcal{A}^0 \subset \mathcal{A}$, $\mathcal{H}(\mathcal{A}) \subset \mathcal{A}$, and the edge matrix $A_E = [A_E(e, f)]_{e, f \in E^1}$ of E :

$$(42) \quad A_E(e, f) := \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0, & \text{otherwise,} \end{cases}$$

can be recovered from the system $(\mathcal{A}, \mathcal{H})$ since

$$A_E(e, f) = 1 \quad \text{if and only if} \quad \mathcal{H}(S_e S_e^*) S_f S_f^* \neq 0.$$

Moreover, we have

Proposition 4.2. *The minimal C^* -algebra \mathcal{B} containing \mathcal{A} and such that $\mathcal{V}(\mathcal{B}) \subset \mathcal{B}$ coincides with the AF-core (34):*

$$\mathcal{B} := C^* \left(\bigcup_{n=0}^{\infty} \mathcal{V}^n(\mathcal{A}) \right) = \mathcal{F}_E.$$

Furthermore, $C^* \left(\bigcup_{n=0}^{\infty} \mathcal{V}^n(\mathcal{A}^0) \right) = \mathcal{F}_E$ if and only if E has no multiedges, and in general there is a natural isomorphism

$$C^* \left(\bigcup_{n=0}^{\infty} \mathcal{V}^n(\mathcal{A}^0) \right) \cong \mathcal{F}_{\tilde{E}}$$

where $\tilde{E} = (E^0, \tilde{E}^1, \tilde{r}, \tilde{s})$, is the graph obtained from E by passing to quotients with respect to the equivalence relation \sim on E^1 : $e \sim f \iff r(e) = r(f), s(e) = s(f)$.

Proof. For the first part of assertion we note that $\mathcal{B} = C^* \left(\bigcup_{n=0}^{\infty} \mathcal{V}^n(\mathcal{A}) \right) \subset \mathcal{F}_E$ and to show the opposite inclusion note that (39) and (33) imply

$$S_{e\mu} S_{f\nu}^* = \sqrt{n_{s(\mu)} n_{s(\nu)}} S_e S_e^* \mathcal{V}(S_{\mu} S_{\nu}^*) S_f S_f^*, \quad e, f \in E^1, \mu, \nu \in E^N, N \in \mathbb{N}.$$

Using the above equality one sees that $\mathcal{F}_1 \subset \mathcal{B}$ and by induction $\mathcal{F}_N \subset \mathcal{B}$ for all $N \in \mathbb{N}$, that is $\mathcal{B} = \mathcal{F}_E$. For the second part denote by $\tilde{\mathcal{F}}_N(v)$ and $\tilde{\mathcal{F}}_N$ the algebras associated to the quotient graph \tilde{E} via (35) and (36) respectively. For each sequence of vertices v_1, \dots, v_n define the set of all paths that could be realized on that sequence, that is we put

$$E_{v_1, \dots, v_n} := \{\mu = (\mu_1, \dots, \mu_{n-1}) \in E^{n-1} : s(\mu_i) = v_{i-1}, r(\mu_i) = v_i, i > 1\}.$$

With each such two nonempty sets we associate the operator

$$m_{v_1, \dots, v_n; u_1, \dots, u_n} := \sum_{\substack{\mu \in E_{v_1, \dots, v_n} \\ \nu \in E_{u_1, \dots, u_n}}} \frac{S_{\mu} S_{\nu}^*}{\sqrt{|E_{v_1, \dots, v_n}| \cdot |E_{u_1, \dots, u_n}|}}.$$

If E_{v_1, \dots, v_n} or E_{u_1, \dots, u_n} is empty, put $m_{v_1, \dots, v_n; u_1, \dots, u_n} := 0$. The important feature of these operators is that for the fixed v the nonzero elements $m_{v_1, \dots, v; u_1, \dots, v}$ form the matrix units which generate a copy of $\tilde{\mathcal{F}}_n(v)$. Let us also note that $|E_{v_1, \dots, v_n}| \cdot |E_{v_n, \dots, v_{n+m}}| = |E_{v_1, \dots, v_{n+m}}|$ and

$$m_{v_1, \dots, v_{n+1}; u_1, \dots, u_{n+1}} = \frac{\sqrt{n_{v_2} n_{u_2}}}{\sqrt{|E_{v_1, v_2}| \cdot |E_{u_1, u_2}|}} P_{v_1} \mathcal{V}(m_{v_2, \dots, v_{n+1}; u_2, \dots, u_{n+1}}) P_{u_1},$$

whenever the left hand side is nonzero. Using the above relation one verifies by induction that $C^*(\bigcup_{i=0}^n \mathcal{V}^i(\mathcal{A}_0)) = \text{span}\{m_{v_1, \dots, v_n; u_1, \dots, u_n} : v_i, u_j \in E^0\}$. Therefore we have a natural isomorphism $C^*(\bigcup_{i=0}^n \mathcal{V}^i(\mathcal{A}_0)) \cong \tilde{\mathcal{F}}_N$ and consequently $C^*(\bigcup_{n=0}^\infty \mathcal{V}^n(\mathcal{A}_0))$ is isomorphic to $\mathcal{F}_{\tilde{E}}$. If $E \neq \tilde{E}$, that is there are two different edges $e, f \in E^1$ such that $r(e) = r(f)$ and $s(e) = s(f)$, then neither $S_e S_e^*$ nor $S_f S_f^*$ belong to $C^*(\bigcup_{n=0}^\infty \mathcal{V}^n(\mathcal{A}_0))$ that is $\mathcal{F}_E \neq C^*(\bigcup_{n=0}^\infty \mathcal{V}^n(\mathcal{A}_0))$. \square

Now we may use Proposition 2.9 or [LO04, Thm. 3.11] to determine when the interaction $(\mathcal{V}, \mathcal{H})$ is a C^* -dynamical system. In particular, this is always the case when E has no sources.

Proposition 4.3. *Let $(\mathcal{V}, \mathcal{H})$ be the interaction associated to E . The following conditions are equivalent:*

- i) $(\mathcal{V}, \mathcal{H})$ is a C^* -dynamical system,
- ii) every two paths that starts in sources and ends in the same vertex have the same length,
- iii) $\ker \mathcal{V}$ is an ideal in \mathcal{F}_E .

Proof. We recall that $\mathcal{H}(1) = \sum_{v \in r(E^1)} P_v$. In view of the equivalence i) \Leftrightarrow ii) in Proposition 2.9 the equivalence i) \Leftrightarrow ii) in the present assertion follows from the relations

$$\mathcal{H}(1)S_\mu S_\nu^* = \begin{cases} 0, & \text{if } s(\mu) \notin r(E^1) \\ S_\mu S_\nu^*, & \text{otherwise} \end{cases}, \quad S_\mu S_\nu^* \mathcal{H}(1) = \begin{cases} 0, & \text{if } s(\nu) \notin r(E^1) \\ S_\mu S_\nu^*, & \text{otherwise} \end{cases}$$

i) to iii) which hold for arbitrary $S_\mu S_\nu^*$ where $|\mu| = |\nu|$. The implication i) \Rightarrow iii) is obvious. To see iii) \Rightarrow ii) note that

$$\ker \mathcal{V} = \overline{\text{span}}\{S_\mu S_\nu^* : s(\mu) \notin r(E^1) \text{ or } s(\nu) \notin r(E^1), |\mu| = |\nu| \in \mathbb{N}\}$$

and if we assume that $\mu, \nu \in E^n$, $n \in \mathbb{N}$, are such that $r(\mu) = r(\nu)$, $s(\mu) \notin r(E^1)$ and $s(\nu) \in r(E^1)$, then

$$S_\mu S_\nu^* \in \ker \mathcal{V} \quad \text{but} \quad (S_\nu S_\mu^*)(S_\mu S_\nu^*) = S_\nu S_\nu^* \notin \ker \mathcal{V}.$$

\square

A natural question to ask is when \mathcal{H} is multiplicative. We rush to say that $\ker \mathcal{H} = \text{span}\{P_v : v \text{ is a sink}\}$ is always an ideal in \mathcal{F}_E , however, the pair $(\mathcal{H}, \mathcal{V})$ is hardly ever a C^* -dynamical system.

Proposition 4.4. *The pair $(\mathcal{H}, \mathcal{V})$ where $(\mathcal{V}, \mathcal{H})$ is the interaction associated to E is a C^* -dynamical system if and only if the mapping $r : E^1 \rightarrow E^0$ is injective.*

Proof. By Proposition 2.9 multiplicativity of \mathcal{H} is equivalent to $\mathcal{V}(1)$ being a central element in \mathcal{F}_E . For all $v \in E^0$, $g, h \in E^1$, $\mu, \nu \in E^n$, $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{V}(1)P_v &= \sum_{\substack{f \in s^{-1}(v) \\ e \in r^{-1}(r(f))}} \frac{1}{n_{r(f)}} S_e S_f^*, & P_v \mathcal{V}(1) &= \sum_{\substack{e \in s^{-1}(v) \\ f \in r^{-1}(r(e))}} \frac{1}{n_{r(e)}} S_e S_f^*, \\ \mathcal{V}(1)S_{g\mu} S_{h\nu}^* &= \frac{1}{n_{s(\mu)}} \sum_{e \in r^{-1}(s(\mu))} S_{e\mu} S_{h\nu}^*, & S_{g\mu} S_{h\nu}^* \mathcal{V}(1) &= \frac{1}{n_{r(g)}} \sum_{f \in r^{-1}(s(\nu))} S_{g\mu} S_{f\nu}^*. \end{aligned}$$

In particular, if the mapping $r : E^1 \rightarrow E^0$ is injective one sees that $\mathcal{V}(1)$ belongs to the center of \mathcal{F}_E . Conversely, if we assume that $\mathcal{V}(1)$ is a central element, then

for all $g, h \in E^1$ such that $r(g) = r(h)$ the equality $\mathcal{V}(1)S_g S_h^* = S_g S_h^* \mathcal{V}(1)$ implies that $g = h$, that is $r : E^1 \rightarrow E^0$ is injective. \square

We note that the equivalent conditions of Proposition 4.4 ensures that any S given by (37) is a power partial isometry, e.g. all its powers are partial isometries. This is not trivial, in particular Halmos and Wallen in [HW] presented a method of explicit construction of an operator S such that the distribution of values of n for which S^n is or is not a partial isometry is arbitrary. En passant we discover similar construction based on graphs.

Proposition 4.5. *Let S be the operator given by (37) for a certain Cuntz-Krieger E -family and let $n > 1$. The following conditions are equivalent*

- i) *operator S^n is partial isometry*
- ii) *n -th power of the left quasi-stochastic matrix $P = \{p_{v,w}\}_{v,w \in E^0}$ given by (41) is left quasi-stochastic.*
- iii) *the lengths of all the paths that start in sources and have the common range are either strictly smaller than n or not smaller than n .*
- iv) *there is no vertex which is the range of two paths such that one has length n and the other starts in a source and has length $k < n$.*

The operator S^n , , where S is given by (37) for a certain Cuntz-Krieger E -family, is a partial isometry if and only if

Proof. The equivalence iii) \Leftrightarrow iv) is straightforward. To see the equivalence i) \Leftrightarrow ii) note that S^n is a partial isometry if and only if $S^{*n} S^n = \mathcal{H}^n(1)$ is an orthogonal projection. Moreover, since $\mathcal{H}(P_v) = \sum_{w \in E^0} p_{v,w} P_w$ we get that

$$\mathcal{H}^n(1) = \sum_{v_0, \dots, v_n \in E^0} p_{v_0, v_1} \cdot p_{v_1, v_2} \cdot \dots \cdot p_{v_{n-1}, v_n} P_{v_n} = \sum_{v, w \in E^0} p_{v,w}^{(n)} P_w$$

where $P^n = \{p_{v,w}^{(n)}\}_{v,w \in E^0}$ stands for the n -th power of P . By orthogonality of projections P_w , it follows that $\mathcal{H}^n(1)$ is a projection iff $\sum_{v \in E^0} p_{v,w}^{(n)} \in \{0, 1\}$ for all $w \in E^0$, that is iff P^n is a left quasi-stochastic matrix. This proves i) \Leftrightarrow ii). To show ii) \Leftrightarrow iv) note that the condition $\sum_{v \in E^0} p_{v,w}^{(n)} > 0$ is equivalent to the existence of $\mu \in E^n$ such that $w = r(\mu)$. Let us then consider $w \in E^0$ such that $\sum_{v \in E^0} p_{v,w}^{(n)} > 0$. We claim that the equality $\sum_{v \in E^0} p_{v,w}^{(n)} = 1$ is equivalent to the following implication

$$(43) \quad \forall_{k=1, \dots, n-1} \forall_{v_{n-k} \in E^0} p_{v_{n-k}, w}^{(k)} \neq 0 \implies \sum_{v_0 \in E^0} p_{v_0, v_{n-k}}^{(n-k)} = 1.$$

Indeed, if we assume (43) and apply it for $k = 1$ we get

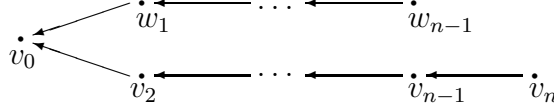
$$\sum_{v \in E^0} p_{v,w}^{(n)} = \sum_{v_0, v_{n-1} \in E^0} p_{v_0, v_{n-1}}^{(n-1)} p_{v_{n-1}, w}^{(1)} = \sum_{v \in E^0} p_{v, w}^{(1)} = 1.$$

Conversely, if we assume that $p_{v_{n-k}, w}^{(k)} \neq 0$ and $\sum_{v_0 \in E^0} p_{v_0, v_{n-k}}^{(n-k)} < 1$, for a certain $k = 1, \dots, n-1$, then

$$\sum_{v \in E^0} p_{v,w}^{(n)} = \sum_{v_0, v_{n-k} \in E^0} p_{v_0, v_{n-k}}^{(n-k)} p_{v_{n-k}, w}^{(k)} < \sum_{v_{n-k} \in E^0} p_{v_{n-k}, w}^{(k)} \leq 1,$$

and thus our claim is proved. Clearly, the condition (43) in terms of graphs means that there is no path starting in a source of length $k < n$ and whose range is w . Hence by arbitrariness of w we get ii) \Leftrightarrow iv). \square

Example 4.6. For any $n > 1$ the partial isometry associated S to the following graph



has a property that the only power of S which is not a partial isometry is the n -th one. Hence by considering a disjoint sum of the above graphs for a chosen sequence of natural numbers $1 < n_1 < n_2 < \dots < n_m$ one obtains a partial isometries S whose k -th power is a partial isometry iff $k \neq n_i$, $i = 1, \dots, m$.

We may use Proposition 4.5 to prolong the list of equivalents in Proposition 4.4.

Corollary 4.7. *Let $(\mathcal{V}, \mathcal{H})$ be the interaction associated to E . The following conditions are equivalent:*

- i) $(\mathcal{V}, \mathcal{H})$ is a C^* -dynamical system,
- ii) every power of the left quasi-stochastic matrix $P = \{p_{v,w}\}_{v,w \in E^0}$ is left quasi-stochastic
- iii) any operator S given by (37) for a certain Cuntz-Krieger E -family is a power partial isometry.

The main conclusion of the present subsection is

Theorem 4.8. *The association to a Cuntz-Krieger E -family a natural faithful representation of \mathcal{F}_E and the partial isometry (39), yields a one-to-one correspondence between Cuntz-Krieger E -families and faithful covariant representations of the interaction $(\mathcal{V}, \mathcal{H})$ associated to E .*

In particular, we have a natural isomorphism

$$C^*(E) \cong \mathcal{F}_E \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$$

where $\mathcal{F}_E \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z}$ is the crossed-product discussed in subsection 2.2, and if the equivalent conditions in Proposition 4.4 hold, then $C^*(E)$ realizes as the crossed product introduced in [ABL].

Proof. We have already observed that any Cuntz-Krieger E -family gives rise via (37) to a covariant representations of $(\mathcal{V}, \mathcal{H})$ where we identify the core C^* -algebra \mathcal{F}_E with $\overline{\text{span}} \{S_\mu S_\nu^* : |\mu| = |\nu|\}$. Conversely, if $\mathcal{F}_E = \overline{\text{span}} \{S_\mu S_\nu^* : |\mu| = |\nu|\}$ is realized via a Cuntz-Krieger E -family $\{P_v : v \in E^0\}$, $\{S_e : e \in E^1\}$, and (π, S) is a covariant representation of the interaction $(\mathcal{V}, \mathcal{H})$, then using (39) and (40) one shows that

$$\tilde{P}_v := \pi(P_v), \quad \tilde{S}_e := \sqrt{n_{r(e)}} \pi(S_e S_e^*) S \pi(P_{r(e)}), \quad v \in E^0, e \in E^1,$$

is a Cuntz-Krieger E -family such that $S = \sum_{e \in E^1} \frac{\tilde{S}_e}{\sqrt{n_{r(e)}}}$. Indeed, for $e \in E^1$ we have

$$\tilde{S}_e^* \tilde{S}_e = n_{r(e)} \pi(P_{r(e)}) \pi(\mathcal{H}(S_e S_e^*)) \pi(P_{r(e)}) = \pi(P_{r(e)}) = \tilde{P}_{r(e)},$$

and for $v \in s(E^1)$

$$\begin{aligned}
\sum_{e \in s^{-1}(v)} \tilde{S}_e \tilde{S}_e^* &= \sum_{e \in s^{-1}(v)} n_{r(e)} \pi(S_e S_e^*) \pi(\mathcal{V}(P_{r(e)})) \pi(S_e S_e^*) \\
&= \sum_{e \in s^{-1}(v), e_1, e_2 \in r^{-1}(r(e))} \pi(S_e S_e^* (S_{e_1} S_{e_2}^*) S_e S_e^*) = \sum_{e \in s^{-1}(v)} \pi(S_e S_e^*) \\
&= \pi(P_{r(e)}) = \tilde{P}_{r(v)}.
\end{aligned}$$

To see that $S = \sum_{e \in E_1} \frac{\tilde{S}_e}{\sqrt{n_{r(e)}}}$ note that the initial subspace of S is a sum of orthogonal images of the projections $\pi(P_{r(e)})$ (we have $S^* S = \sum_{e \in E^1} \pi(P_{r(e)})$). Moreover, since for each $v \in E^0$ such that $r^{-1}(v) \neq \emptyset$ we have

$$\begin{aligned}
\left(\sum_{e \in r^{-1}(v)} \pi(S_e S_e^*) \right) S \pi(P_v) S^* &= \sum_{e \in r^{-1}(v)} \pi(S_e S_e^* \mathcal{V}(P_v)) = \sum_{e, e_1, e_2 \in r^{-1}(v)} \frac{\pi(S_e S_e^* S_{e_1} S_{e_2}^*)}{n_v} \\
&= \sum_{e_1, e_2 \in r^{-1}(v)} \frac{\pi(S_{e_1} S_{e_2}^*)}{n_v} = \pi(\mathcal{V}(P_v)) = S \pi(P_v) S^*,
\end{aligned}$$

it follows that the final space of the partial isometry $S \pi(P_v)$ decomposes into the orthogonal sum of range spaces of projections $\pi(S_e S_e^*)$, $e \in r^{-1}(v)$. Thus

$$\sum_{e \in E_1} \frac{\tilde{S}_e}{\sqrt{n_{r(e)}}} = \sum_{e \in E_1} \pi(S_e S_e^*) S \pi(P_{r(e)}) = S.$$

□

Remark 4.9. If E has no sources, then S given by (37) is an isometry and $\alpha = \mathcal{V}$ is a monomorphism (with hereditary range). In this case $C^*(E)$ coincides with various crossed products that involve isometries, cf. [ABL]. In particular, recently (and independently to author) Huef and Raeburn [HR] proved a version of Theorem 4.8 for infinite graphs by showing that $C^*(E)$ identifies with Stacey's (multiplicity-one) crossed product of \mathcal{F}_E by α in the case E is infinite, without sources and such that the numbers $n_v = |r^{-1}(v)|$ are finite. However, if E has sources, then \mathcal{F}_E does not embed into $\mathcal{F}_E \rtimes_{\alpha} \mathbb{Z}$ for isometric crossed products (one has to reduce relations to "make" α injective, cf. e.g. [KL] or [Kwa, Ex. 6.22]) and thus $C^*(E)$ cannot be realized as a Stacey crossed product of \mathcal{F}_E by α in this case.

4.3. Topological Markov chains as diagonalizations of the interactions.

Before we pass to the essential theme of our analysis, we briefly discuss the relationship between the presentation of $C^*(E)$ as $\mathcal{F}_E \rtimes_{\alpha} \mathbb{Z}$ and a somewhat classical approach based on topological Markov chains. This could be of interest and shall serve as an instructive model example.

A crossed product approach to $C^*(E)$, already indicated by Cuntz and Krieger [CK80], was formalized by R. Exel in [Exe03]. In a sense this construction relies upon a "diagonalization" of the interaction $(\mathcal{V}, \mathcal{H})$ on \mathcal{F}_E . Indeed, it is well known that the C^* -algebra

$$\mathcal{D}_E := \overline{\text{span}} \{ S_{\mu} S_{\mu}^* : |\mu| = 0, 1, \dots \}$$

is a masa in \mathcal{F}_E (a maximal abelian self-adjoint subalgebra). The operator $\mathcal{H}(\cdot) = S^*(\cdot)S$, where S is given by (37), invariants \mathcal{D}_E and the smallest C^* -algebra containing \mathcal{D}_E and preserved under $\mathcal{V}(\cdot) = S(\cdot)S^*$ is $\mathcal{B} = \mathcal{F}_E$. On the other hand the map $\phi_E : C^*(E) \rightarrow C^*(E)$ given by

$$\phi_E(x) = \sum_{e \in E^1} S_e x S_e^*$$

is a completely positive contraction that preserves not only the AF-core \mathcal{F}_E but also \mathcal{D}_E . Significantly, $\mathcal{D}_E = C^*(\bigcup_{n=0}^{\infty} \phi_E^n(\mathcal{A}))$ is the minimal C^* -algebra containing \mathcal{A} and preserved by ϕ_E . Moreover, ϕ_E restricted to \mathcal{D}_E is an endomorphism whose dual map is a Markov shift σ_E , and \mathcal{H} restricted to \mathcal{D}_E is a classical transfer operator for σ_E . More precisely, similarly like [CK80, Prop. 2.5], one obtains the following

Proposition 4.10. *Algebra \mathcal{D}_E is isomorphic to the algebra of functions $C(\Omega_E)$ on the space*

$$\Omega_E = \bigcup_{N=0}^{\infty} E_{sinks}^N \cup E^{\infty}$$

where for $N \geq 1$,

$$E_{sinks}^N = \{(\mu_1, \dots, \mu_N, 0, \dots) : (\mu_1, \dots, \mu_N) \in E^N \text{ and } r(\mu_N) \text{ is a sink}\}$$

form a discrete open subspace of Ω_E and $\Omega_E \setminus E_{sinks}^0$ is equipped with the product topology inherited from $\prod_{n=1}^{\infty} (E^1 \cup \{0\})$, where $E^1 \cup \{0\}$ is a discrete space. Under the identification $\mathcal{D}_E = C(\Omega_E)$ the mapping dual to $\phi_E : \mathcal{D}_E \rightarrow \mathcal{D}_E$ is a shift σ_E defined on $\Omega_E \setminus E_{sinks}^0$ via the formula

$$(44) \quad \sigma_E(\mu_1, \mu_2, \mu_3 \dots) = (\mu_2, \mu_3 \dots) \quad \text{for } (\mu_1, \mu_2 \dots) \in \bigcup_{N=2}^{\infty} E_{sinks}^N \cup E^{\infty},$$

and $\sigma_E(\mu_1, 0, 0, \dots) = r(\mu_1)$ for $(\mu_1, 0, 0 \dots) \in E_{sinks}^1$. The operator \mathcal{H} acts on $f \in \mathcal{D}_E = C(\Omega_E)$ as follows

$$\mathcal{H}(f)(\mu) = \begin{cases} \frac{1}{|\sigma_E^{-1}(\mu)|} \sum_{\nu \in \sigma_E^{-1}(\mu)} f(\nu), & \text{if } \mu \in \sigma_E(\Omega_E), \\ 0, & \text{if } \mu \notin \sigma_E(\Omega_E), \end{cases}$$

where $\sigma_E(\Omega_E) = \{\mu \in \Omega_E : s(\mu) \text{ is not a source}\}$ and $|\sigma_E^{-1}(\mu)| = |r^{-1}(s(\mu))|$.

Proof. One checks that (the necessarily unique) mapping $\Phi : C(\Omega_E) \rightarrow \mathcal{D}_E$ such that

$$\chi_{\{v\}} \longmapsto P_v, \quad \text{for } v \in E_{sinks}^0,$$

$$\chi_{\{\nu=(\nu_1, \dots) \in \Omega_E : \nu_i = \mu_i, i=1, \dots, n\}} \longmapsto S_{\mu} S_{\mu}^* \quad \text{for } \mu = (\mu_1, \dots, \mu_n) \in E^n,$$

is a well defined isomorphism which intertwines the composition operator with the shift σ_E and the endomorphism $\phi_E : \mathcal{D}_E \rightarrow \mathcal{D}_E$. \square

Remark 4.11. The pair (ϕ_E, \mathcal{H}) is a C^* -dynamical system, however its powers might not be. More precisely, the pair $(\phi_E^n, \mathcal{H}^n)$, $n > 1$ is a C^* -dynamical system if and only if the equivalent conditions in Proposition 4.5 hold, cf. Remark 2.4.

In the case E has no sinks the system (Ω_E, σ_E) reduces to the standard Markov chain (E^∞, σ_E) and by [Exe03, Thm. 6.2], $C^*(E)$ is isomorphic to Exel's crossed product $C(E^\infty) \rtimes_{\phi_E, \mathcal{L}} \mathbb{N}$, where $\mathcal{L} = \mathcal{H} : C(E^\infty) \rightarrow C(E^\infty)$ is implemented by a universal partial isometry S and $C^*(E) = C^*(\mathcal{D}_E, S)$. In fact, Exel's partial isometry S satisfies (37) and we have

$$C^*(\mathcal{D}_E, S) = C^*(\mathcal{F}_E, S),$$

that is $\mathcal{F}_E \rtimes_{(\mathcal{V}, \mathcal{H})} \mathbb{Z} = C(E^\infty) \rtimes_{\phi_E, \mathcal{L}} \mathbb{N}$. It seems that the principle role in Exel's approach plays the transfer operator \mathcal{L} not the mapping ϕ_E . For instance, there are uncountably many transfer operators for σ_E while there is only one mapping (endomorphism) for which $\mathcal{L} : \mathcal{D}_E \rightarrow \mathcal{D}_E$ is a transfer operator. Also, in view of our presentation it is natural to think of $C^*(E)$ as a crossed product of \mathcal{A} by \mathcal{L} . Concluding, we have an ascending sequence of algebras

$$\mathcal{A} \subset \mathcal{D}_E \subset \mathcal{F}_E \subset C^*(E)$$

each of which could serve as a starting point for the construction of $C^*(E)$. However, \mathcal{F}_E is distinguished as the core algebra and perhaps it is reasonable to consider the interaction $(\mathcal{V}, \mathcal{H})$ not the linear map $\phi_E : \mathcal{F}_E \rightarrow \mathcal{F}_E$ as an appropriate non-commutative counterpart of the Markov shift. The simplest case of Bernoulli shifts is a very visible support of this point of view.

Example 4.12. Let us consider the case of Cuntz algebra \mathcal{O}_n , i.e. we suppose that $|E^0| = 1$, $|E^1| = n$ and thus A_E is $n \times n$ matrix with all entries equal to 1. Then the interaction $(\mathcal{V}, \mathcal{H}) = (\alpha, \mathcal{L})$ is a C^* -dynamical system, \mathcal{F}_E is the Glimm's UHF algebra and $\mathcal{F}_E = M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \dots$. Using this multiplicative notation we have

$$\alpha(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes \dots) = \frac{1}{n} A_E \otimes a^{(1)} \otimes a^{(2)} \otimes \dots,$$

$$\mathcal{L}(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes \dots) = L(a^{(1)}) \cdot a^{(2)} \otimes a^{(3)} \otimes a^{(4)} \otimes \dots$$

where $L : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ is the standard tracial state: $L(a) = \frac{1}{n} \sum_{i,j=1}^n a_{ij}$. Visibly, α and \mathcal{L} could be treated as a forward and backward shifts on \mathcal{F}_E . Moreover, they induce shifts on the state space $S(\mathcal{F}_E)$ of \mathcal{F}_E . To be more precise, if $\omega \in S(\mathcal{F}_E)$, then $\mathcal{L}^*(\omega) = \omega \circ \mathcal{L}$ is the state and if $\omega = \omega^{(1)} \otimes \omega^{(2)} \otimes \dots$ we have

$$\mathcal{L}^*(\omega^{(1)} \otimes \omega^{(2)} \otimes \dots) = L \otimes \omega^{(1)} \otimes \omega^{(2)} \otimes \dots$$

The composition $\omega \circ \alpha$ is a multiple of a state and denoting it by $\alpha^*(\omega)$ we have

$$\alpha^*(\omega^{(1)} \otimes \omega^{(2)} \otimes \dots) = \begin{cases} \omega^{(2)} \otimes \omega^{(3)} \otimes \dots, & \text{if } \omega^{(1)}(A_E) \neq 0, \\ 0, & \text{if } \omega^{(1)}(A_E) = 0. \end{cases}$$

In particular, the mappings α and \mathcal{L} give rise to mutually inverse homeomorphisms between the pure states spaces: $\alpha^* : P(\alpha(\mathcal{F}_E)) \rightarrow P(\mathcal{F}_E)$ and $\mathcal{L}^* : P(\mathcal{F}_E) \rightarrow P(\alpha(\mathcal{F}_E))$.

4.4. Description of the dynamical system dual to $(\mathcal{V}, \mathcal{H})$. We obtain a satisfactory picture of the system $(\widehat{\mathcal{F}_E}, \widehat{\mathcal{V}})$ dual to the interaction $(\mathcal{V}, \mathcal{H})$ associated to E , cf. Remark 2.22, by showing that the topological Markov chain described in Proposition 4.10 factors through to a subsystem of $(\widehat{\mathcal{F}_E}, \widehat{\mathcal{V}})$.

Let us note that the infinite direct sum $\bigoplus_{N=0}^{\infty} \bigoplus_{w \in E_{sinks}^0} \mathcal{F}_N(w)$, yields an ideal I_{sinks} in \mathcal{F}_E , cf. (36), generated by the projections P_w , $w \in E_{sinks}^0$. We rewrite it in the following form

$$I_{sinks} = \bigoplus_{N \in \mathbb{N}} G_N, \quad \text{where} \quad G_N := \left(\bigoplus_{w \text{ is a sink}} \mathcal{F}_N(w) \right).$$

Then the algebra $C(E_{sinks}^N)$ identifies with a masa in G_N , $N \in \mathbb{N}$. The spectrum of I_{sinks} is a discrete subspace of $\widehat{\mathcal{F}_E}$. For each sink w and $N \in \mathbb{N}$ the space $\mathcal{F}_N(w)$ is a factor in \mathcal{F}_E and thus if $\mathcal{F}_N(w) \neq \{0\}$ we may associate to it a unique up to equivalence irreducible representation $\pi_{w,N}$ of \mathcal{F}_E such that $\ker \pi_{w,N} \cap \mathcal{F}_N(w) = \{0\}$. Consequently

$$\widehat{G}_N = \{\pi_{w,N} : \text{there is } \mu \in E_{sinks}^N \text{ such that } r(\mu) = w\}$$

(we abuse the notation and treat E_{sinks}^N as a set of paths from E^N which end in sinks). The complement of $\widehat{I}_{sinks} = \bigcup_{N=0}^{\infty} \widehat{G}_N$ in $\widehat{\mathcal{F}_E}$ is a closed set which we identify with the spectrum of the quotient algebra

$$G_{\infty} := \mathcal{F}/I_{sinks}.$$

We describe elements in \widehat{G}_{∞} that are AF-analogues of representations arising from product states on UHF-algebras, cf. Example 4.12. We shall use an equivalence relation on the set of infinite paths E^{∞} defined as follows

$$\mu \sim \nu \iff \text{there exists } N \text{ such that } (\mu_N, \mu_{N+1}, \dots) = (\nu_N, \nu_{N+1}, \dots),$$

and denote by $W(\mu)$ the equivalence class for μ ; in [CK80], $W(\mu)$ is referred to as an *unstable manifold* of μ .

Proposition 4.13. *For any infinite path $\mu \in E^{\infty}$ the formula*

$$(45) \quad \omega_{\mu}(S_{\nu} S_{\eta}^*) = \begin{cases} 1 & \nu = \eta = (\mu_1, \dots, \mu_n) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \nu, \eta \in E^n$$

determines a pure state $\omega_{\mu} : \mathcal{F}_E \rightarrow \mathbb{C}$ (a pure extension of the point evaluation δ_{μ} acting on the masa $\mathcal{D}_E = C(\Omega_E)$). Moreover, denoting by π_{μ} the GNS-representation associated to ω_{μ} we have $\pi_{\mu} \in \widehat{G}_{\infty}$ and

- i) the complement of the subdiagram of the Bratteli diagram $\Lambda(E)$ corresponding to $\ker \pi_{\mu}$ consists of all vertices in $\Lambda(E)$ that form paths from $W(\mu)$ (i.e. all ancestors of the vertices that form μ).
- ii) representations π_{μ} and π_{ν} are unitarily equivalent if and only if $\mu \sim \nu$.

Proof. The functional ω_{μ} is a pure state on each \mathcal{F}_k , $k \in \mathbb{N}$, and thus its inductive limit is also a pure state, cf. [Bra72, 4.16]. Item i) is clear by the form of primitive ideal subdiagrams, see [Bra72, 3.8]. Item ii) follows from [Bra72, 4.5]. \square

In view of Proposition 4.2 the following statement could be considered as an analogue of Theorem 3.2 – a dual description of the reversible extension $(\mathcal{B}, \mathcal{H})$ of the system $(\mathcal{A}, \mathcal{H})$.

Theorem 4.14. *Under the above notation the space $\widehat{\mathcal{F}}_E$ admits the following decomposition into disjoint sets*

$$\widehat{\mathcal{F}}_E = \bigcup_{N=0}^{\infty} \widehat{G}_N \cup \widehat{G}_{\infty}$$

where \widehat{G}_N are open discrete sets and \widehat{G}_{∞} is a closed subset of $\widehat{\mathcal{F}}_E$ (neighbourhoods of points in \widehat{G}_{∞} depend on the structure of E). The set

$$\Delta = \widehat{\mathcal{F}}_E \setminus \widehat{G}_0$$

is the domain of $\widehat{\mathcal{V}}$ which acts on the corresponding representations as follows:

$$\widehat{\mathcal{V}}(\pi_{(\mu_1, \mu_2, \mu_3, \dots)}) = \pi_{(\mu_2, \mu_3, \dots)}, \quad \text{for } (\mu_1, \mu_2, \mu_3, \dots) \in E^{\infty},$$

$$\widehat{\mathcal{V}}(\pi_{w,N}) = \pi_{w,N-1}, \quad \text{for } w = r(\mu) \text{ where } \mu \in E_{sinks}^N, \quad N \geq 1.$$

In particular, $\pi_{w,N} \in \widehat{\mathcal{V}}(\Delta)$ iff there is $\mu \in E_{sinks}^{N+1}$ such that $r(\mu) = w$, and then $\widehat{\mathcal{H}}(\pi_{w,N}) = \pi_{w,N+1}$. Similarly, $\pi_{\mu} \in \widehat{\mathcal{V}}(\Delta)$ iff there is $\nu \in W(\mu)$ such that $s(\nu)$ is not a source, that is there exists $\nu \sim \mu$ and $\nu_0 \in E^1$ such that $(\nu_0, \nu_1, \nu_2, \dots) \in E^{\infty}$, and then $\widehat{\mathcal{H}}(\pi_{\mu}) = \pi_{(\nu_0, \nu_1, \nu_2, \dots)}$.

Proof. The first part of assertion follows immediately from construction of the sets \widehat{G}_N , \widehat{G}_{∞} and from (36). To see that $\widehat{\mathcal{V}}(\widehat{\mathcal{F}}_E) = \{\pi \in \widehat{\mathcal{F}}_E : \pi(\mathcal{V}(1)) \neq 0\}$ coincides with $\Delta = \widehat{\mathcal{F}}_E \setminus \widehat{G}_0$ let $\pi \in \widehat{\mathcal{F}}_E$ and note that

$$\pi(\mathcal{V}(1)) = 0 \iff \forall_{v \in s(E^1)} \pi(P_v) = \pi(P_v \mathcal{V}(1)) = 0 \iff \exists_{w \in E_{sinks}^0} \pi \cong \pi_{w,0}.$$

Furthermore, by (39) and (40), for $N \in \mathbb{N}$ we have

$$(46) \quad \mathcal{V}(\mathcal{F}_N(v)) = \mathcal{V}(1) \mathcal{F}_{N+1}(v) \mathcal{V}(1), \quad \mathcal{H}(\mathcal{F}_N(v)) = \begin{cases} \mathcal{F}_{N-1}(v), & N > 0, \\ \sum_{w \in E^0} p_{v,w} \mathcal{F}_0(w) & N = 0. \end{cases}$$

In particular for $N > 0$ we have $\pi_{w,N} \in \Delta$, $\mathcal{F}_{N-1}(v) \subset \mathcal{H}(\mathcal{F}_E)$ and

$$(\pi_{w,N} \circ \mathcal{V})(\mathcal{F}_{N-1}(w)) = \pi_{w,N}(\mathcal{V}(\mathcal{F}_{N-1}(w))) = \pi_{w,N}(\mathcal{V}(1) \mathcal{F}_N(w) \mathcal{V}(1)) \neq 0,$$

hence $\widehat{\mathcal{V}}(\pi_{w,N}) \cong \pi_{w,N-1}$. Let us now fix $\mu = (\mu_1, \mu_2, \mu_3, \dots) \in E^{\infty}$ and let $\pi_{\mu} : \mathcal{F}_E \rightarrow H_{\mu}$ be the representation and $\xi_{\mu} \in H_{\mu}$ the cyclic vector associated to the pure state ω_{μ} given by (45). Firstly, let us note that

$$\omega_{\mu} \circ \mathcal{V} = \frac{1}{n_{r(\mu_1)}} \omega_{\sigma_E(\mu)}.$$

Indeed, for $\nu, \eta \in E^n$, using (39) and (45), one gets

$$\begin{aligned} \omega_{\mu}(\mathcal{V}(S_{\nu} S_{\eta}^*)) &= \begin{cases} \frac{1}{\sqrt{n_{s(\nu)} n_{s(\eta)}}} \sum_{e, f \in E^1} \omega_{\mu}(S_{e\nu} S_{f\eta}^*), & n_{s(\nu)} n_{s(\eta)} \neq 0, \\ 0, & n_{s(\nu)} n_{s(\eta)} = 0, \end{cases} \\ &= \begin{cases} \frac{1}{n_{r(\mu_1)}}, & \nu = \eta = (\mu_2, \dots, \mu_{n+1}) \\ 0 & \text{otherwise} \end{cases} = \frac{1}{n_{r(\mu_1)}} \omega_{\sigma_E(\mu)}(S_{\nu} S_{\eta}^*). \end{aligned}$$

Secondly, $\pi(\mathcal{V}(1))\xi_\mu$ is a cyclic vector for the irreducible representation $\pi_\mu \circ \mathcal{V} : \mathcal{H}(\mathcal{F}_E) \rightarrow \pi(\mathcal{V}(1))H_\mu$, and for the functional $\phi : \mathcal{H}(\mathcal{F}_E) \rightarrow \mathbb{C}$ associated to the normalization of the cyclic vector $\pi(\mathcal{V}(1))\xi_\mu$ for $\pi_\mu \circ \mathcal{V}$ we have

$$\begin{aligned} \phi(a) &= \frac{1}{\|\pi(\mathcal{V}(1))\xi_\mu\|^2} \langle \pi(\mathcal{V}(a))\pi(\mathcal{V}(1))\xi_\mu, \pi(\mathcal{V}(1))\xi_\mu \rangle \\ &= \frac{1}{\omega_\mu(\mathcal{V}(1))} \langle \pi(\mathcal{V}(a))\xi_\mu, \xi_\mu \rangle = n_{r(\mu_1)} \cdot \omega_\mu(\mathcal{V}(a)) = \omega_{\sigma_E(\mu)}(a). \end{aligned}$$

Therefore $\widehat{\mathcal{V}}(\pi_\mu) \cong \pi_{\sigma_E(\mu)}$. The rest now follows. \square

Remark 4.15. If we extend the equivalence relation \sim from the set E^∞ onto the whole space Ω_E defining it for $\mu \in E_{sinks}^N$ as follows

$$\mu \sim \nu \iff \nu \in E_{sinks}^N \text{ and } r(\mu_N) = r(\nu_N),$$

then Theorem 4.14 states that the quotient system $(\Omega_E / \sim, \sigma_E / \sim)$ is a subsystem of $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$ and the relation \sim coincides with the unitary equivalence of GNS-representations associated to pure extensions of the pure states of $\mathcal{D}_E = C(\Omega_E)$.

Remark 4.16. The nontrivial dynamics of the system $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$ takes place in the subsystem $(\widehat{G}_\infty, \widehat{\mathcal{V}})$ and hence it is worth noting that G_∞ is a C^* -algebra arising from a graph which has no sinks. Indeed, the saturation $\overline{E^0}_{sinks}$ of E^0_{sinks} (the minimal saturated set containing E^0_{sinks}) is the hereditary and saturated set corresponding to the ideal I_{sinks} . Hence $I_{sinks} = \mathcal{F}_{\overline{E^0}_{sinks}}$ and

$$G_\infty \cong \mathcal{F}_{E^0_{sinkless}} \quad \text{where} \quad E^0_{sinkless} := E^0 \setminus \overline{E^0}_{sinks}.$$

4.5. Identification of topological freeness and invariant sets for $\widehat{\mathcal{V}}$. The condition (L) presented in [KPR98] requires that every loop in E has an exit. For convenience, by loops we shall mean simple loops, that is paths $\mu = (\mu_1, \dots, \mu_n)$ such that $s(\mu_1) = r(\mu_n)$ and $s(\mu_1) \neq r(\mu_k)$, for $k = 1, \dots, n-1$. A loop μ is said to have an exit if it is connected to a vertex not lying on μ . We shall deduce, using Theorem 4.14, that condition (L) is equivalent to topological freeness of the partial mapping $\widehat{\mathcal{V}}$.

We start with an easier part which shows that our main result is not weaker than the classical Cuntz-Krieger uniqueness theorem, cf. [KPR98], [CK80].

Proposition 4.17. *If every loop in E has an exit, then every nonempty open set in \widehat{G}_∞ contains uncountably many non-periodic points for $\widehat{\mathcal{V}}$.*

In particular, if every loop in E has an exit, then $\widehat{\alpha}$ is topologically free.

Proof. By Remark 4.16 we may assume that $G_\infty = \mathcal{F}_E$, i.e. E has no sinks. Any nonempty open set in $\widehat{\mathcal{F}}_E$ is of the form $\widehat{J} = \{\pi \in \widehat{\mathcal{F}}_E : \ker \pi \not\supseteq J\}$ where J is a non-zero ideal in \mathcal{F}_E . Equivalently, in terms of Bratteli diagrams

$$\widehat{J} = \{\pi \in \widehat{\mathcal{F}}_E : \Lambda(J) \setminus \Lambda(\ker \pi) \neq \emptyset\}$$

where $\Lambda(K)$ stands for the Bratteli diagram of an ideal K in \mathcal{F}_E . Since E is finite, without sinks, every loop in E has an exit and $\Lambda(J)$ contains all its descendants, one can construct uncountably many non-periodic paths $\mu \in E^\infty$ with different unstable manifolds $W(\mu)$ contained in $\Lambda(J)$. Indeed, there must be a vertex v which appears in $\Lambda(J)$ infinitely many times and which is a base point

of two different loops say μ^0 and μ^1 . Writing $\mu^\epsilon = \mu^{\epsilon_1} \mu^{\epsilon_2} \mu^{\epsilon_3} \dots \in E^\infty$ for an infinite sequence $\epsilon = \{\epsilon_i\}_{i=1}^\infty \in \{0, 1\}^{\mathbb{N} \setminus \{0\}}$ of zeros and ones one has $W(\mu^\epsilon) \subset \Lambda(J)$ and $W(\mu^\epsilon) = W(\mu^{\epsilon'})$ if and only if ϵ and ϵ' eventually coincide. There is an uncountable number of non-periodic sequences in $\{0, 1\}^{\mathbb{N} \setminus \{0\}}$ which (pair-wisely) do not eventually coincide and thus, in view of Proposition 4.13 and Theorem 4.14, the paths corresponding to these sequences give rise to the uncountable family of non-equivalent non-periodic representations π_μ belonging to \widehat{J} . \square

Remark 4.18. An extreme case where the above proposition applies is the one considered in Example 4.12. In this particular example \mathcal{F}_E is simple, $\text{Prim}(\mathcal{F}_E) = \{0\}$ and $\widehat{\mathcal{F}}_E$ is the only nonempty open set in the uncountable space $\widehat{\mathcal{F}}_E$. Hence the system $(\widehat{\mathcal{F}}_E, \widehat{\alpha})$ is topologically free while $(\text{Prim}(\mathcal{F}_E), \widehat{\alpha})$ is not.

Suppose now that μ is a loop in E . Let $\mu_\infty \in E^\infty$ be the path obtained by the infinite concatenation of μ and let us treat the unstable manifold $W(\mu_\infty)$ of μ_∞ as a subdiagram of $\Lambda(E)$. Then the complement $\Lambda(E) \setminus W(\mu_\infty)$ is a Bratteli diagram for a primitive ideal in \mathcal{F}_E , which we denote by I_μ . Actually, by Proposition 4.13 i) we have

$$I_\mu = \ker \pi_{\mu_\infty}$$

where π_{μ_∞} is the irreducible representation associated to μ_∞ .

Proposition 4.19. *If the loop μ has no exits, then up to unitary equivalence π_{μ_∞} is the only representation of \mathcal{F}_E whose kernel is I_μ . Moreover, the singleton $\{\pi_{\mu_\infty}\}$ is an open set in $\widehat{\mathcal{F}}_E$.*

Proof. The quotient \mathcal{F}_E/I_μ is an AF-algebra with the diagram $W(\mu_\infty)$. The path μ_∞ treated as a subdiagram of $W(\mu_\infty)$ is hereditary and its saturation $\overline{\mu_\infty}$ yields the ideal \mathcal{K} in \mathcal{F}_E/I_μ . Since μ_∞ has no exits, \mathcal{K} is isomorphic to the ideal of compact operators $\mathcal{K}(H)$ on a Hilbert space H . Therefore, every faithful irreducible representation of \mathcal{F}_E/I_μ is unitarily equivalent to the unique irreducible extension of the identity representation of $\mathcal{K} = \mathcal{K}(H)$. This shows that π_{μ_∞} is determined by its kernel. Moreover, the subdiagram $\overline{\mu_\infty}$ is hereditary and saturated not only in $W(\mu_\infty)$ but also in $\Lambda(E)$. Thus if we let \mathcal{K} stand for the ideal corresponding to $\overline{\mu_\infty}$ in \mathcal{F}_E one obtains

$$\{P \in \text{Prim}(\mathcal{F}_E) : P \not\supseteq \mathcal{K}\} = \{P \in \text{Prim}(\mathcal{F}_E) : \mathcal{K} \cap P = \{0\}\} = \{I_\mu\},$$

that is $\{I_\mu\}$ is open in $\text{Prim}(\mathcal{F}_E)$ and hence $\widehat{\mathcal{K}} = \{\pi_{\mu_\infty}\}$ is open in $\widehat{\mathcal{F}}_E$. \square

Remark 4.20. One sees that the first part of the above assertion holds in a more general setting when the starting object is a loop $\mu = (\mu_0, \dots, \mu_{n-1})$ such that there is no other loop attached to it, i.e. the only simple loop with a base point $r(\mu_k)$ is $(\mu_{k+1 \pmod n}, \dots, \mu_{k-1 \pmod n}, \mu_k)$. Moreover, if μ has no entrance, then π_{μ_∞} is one-dimensional and the singleton $\{\pi_{\mu_\infty}\}$ is closed in $\widehat{\mathcal{F}}_E$.

Combining Propositions 4.17 and 4.19 we do not only characterize the topological freeness of $(\widehat{\mathcal{F}}_E, \widehat{\mathcal{V}})$ but also spot out an interesting dichotomy concerning its core subsystem $(\widehat{G}_\infty, \widehat{\mathcal{V}})$.

Theorem 4.21. *We have the following dynamical dichotomy:*

- i) *either every nonempty open set in \widehat{G}_∞ contains uncountable nonperiodic points for $\widehat{\mathcal{V}}$ (this holds if every loop in E has an exit), or*

- ii) *there are singletons in \widehat{G}_∞ which are open in $\widehat{\mathcal{F}}_E$, consisting of periodic points for \widehat{V} (they correspond to loops without exits)*

In particular, \widehat{V} is topologically free if and only if every loop in E has an exit.

Remark 4.22. In [Kat04] Katsura exchanges the roles of the range and source maps so that graph E satisfies (L) iff every loop in E has an entrance, and thus condition (L) becomes consistent with Definition 3.5. However, there might be a point in differentiating the cases of maps and graphs. For instance, the space \widetilde{M} in Theorem 3.2 arises as a kind of inverse limit of (M, φ) (the initial algebra is extended via α) while \mathcal{F}_E arises as a kind of noncommutative version of a direct limit of (Ω_E, σ_E) (the initial algebra is extended using \mathcal{H}).

The conclusion of the above statement justifies the title of the present paper since we may restate it in the following way.

Theorem 4.23. *For graph C^* -algebras $C^*(E)$ (equipped with standard gauge actions) Theorem 1.6 and Cuntz-Krieger uniqueness theorem are equivalent.*

For the sake of completeness, we end this section briefly discussing how to obtain an ideal lattice description and simplicity criteria for $C^*(E)$ by determining invariant open sets in $\widehat{\mathcal{F}}_E$.

Proposition 4.24. *The map $V \mapsto \widehat{\mathcal{F}_{\Lambda(V)}}$ is a one-to-one correspondence between the hereditary saturated subset of E^0 and open invariant sets for $\widehat{\alpha}$.*

Proof. It suffices to show that the map

$$V \mapsto \Lambda(V)$$

is a one-to-one correspondence between the hereditary saturated subset of E^0 and Bratteli diagrams for ideals in \mathcal{F}_E satisfying (24). This follows from (46). \square

Corollary 4.25. [BPRS00, Thm. 4.1] *We have a one-to-one correspondence between hereditary saturated subset of E^0 and gauge invariant ideals in $C^*(E)$.*

To obtain a version of the above corollary which describes all the ideals of $C^*(E)$ one needs to impose on E to be such that for every hereditary and saturated set $V \subset E^0$ every loop in the subgraph $E \setminus V$ has an exit in $E \setminus V$. Such a property was called condition (K) in [KPRR97] (originally condition (II) in [Cun81]) and its contradiction is equivalent to existence of a vertex in E^0 which is a base point of a precisely one loop, see [BPRS00]. Thus we have

Corollary 4.26. [BPRS00, Thm. 4.4], [KPRR97, Thm 6.6] *Suppose that every vertex in E^0 is either a base point of at least two different loops or does not lie on any loop. Then all ideals in $C^*(E)$ are gauge invariant and in particular there is a lattice isomorphism between the lattice of hereditary and saturated subsets of E^0 and the lattice of ideals in $C^*(E)$.*

Corollary 4.27. [BPRS00, Prop. 5.1], [KPRR97, Cor. 6.8] *If every loop in E has an exit and there are no non-trivial hereditary and saturated subsets of E^0 , then $C^*(E)$ is simple.*

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